1.1 Propositional Logic

<u>Definition</u>: A <u>proposition</u> is a declarative sentence that is either true or false, but not both.

<u>Examples</u>: Are the following propositions?

1. Raleigh is the capital of North Carolina.

   Yes, True statement

2. Cleveland is the capital of Ohio.

   Yes, False statement

3. What day is it?

   No, question not a declarative sentence

4. $x + 1 = 5$

   No, x is a variable so it could be either true or false

5. $1+1=3$

Yes, False statement

6. Chapel Hill is the best city in N.C.

Trick question, best is ill-defined. This could be true or false depending on who you ask. Making well-defined definitions in mathematics is important.

We use letters to denote <u>propositional variables</u> just as letters are used to denote numerical variables. Typically we use the variables $p, q, r,$ and s.

The <u>truth value</u> of a proposition is true, denoted by T, if it is a true proposition. (For example, $1+1=2$.) The truth value of a proposition is false, denoted by F, if it is a false proposition. (For example $1+1=3$).

Propositions that cannot be expressed in terms of simpler propositions are called <u>atomic propositions</u>. The area of logic that deals with propositions is called propositional logic. New propositions, called <u>compound propositions</u>, are formed from existing propositions using <u>logical operators</u>.

<u>Examples</u>:

We do this in regular language all the time.

- My car is 5 years old.
- My car is gray.

Both are atomic propositions and we can combine them to make compound propositions like,

- My car is 5 years old and gray.

or

- My car is not gray.

We will formalize this idea we all use everyday.

<u>Definition 1</u>: Let p be a proposition. The <u>negation</u> of p, denoted by $\neg p$ is the statement

"It is not the case that p"

The proposition $\neg p$ is read "not p". The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

<u>Remark</u>: $\neg p$ is denoted in various ways: $\neg p$, $-p$, Np, $\sim p$, p', $!p$.

<u>Example</u>:

Find the negation of the proposition

- "Michael's PC runs Windows"

"It is not the case Michael's PC runs windows"

or alternatively

"Michael's PC does not run windows"

- "Vandana's computer has at least 32 GB of memory"

"It is not the case that Vandana's computer has at least 32 GB of memory"

"Vandana's computer does not have at least 32 GB of memory"

"Vandana's computer has less than 32 GB of memory"

Truth Tables

These show the truth value of compound propositions given the truth value of the simple propositions

Example for negation

p	$\neg p$
T	F
F	T

These will help for more complicated compound propositions.

Logical operators that form new propositions from two or more propositions are called connectives.

Definition 2: Let p and q be propositions. The conjunction of p and q, denoted $p \wedge q$, is the proposition "p and q". The conjunction $p \wedge q$ is true when both p and q are true and false otherwise.

Definition 3: Let p and q be propositions The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q". The disjunction is false when p and q are false and true otherwise.

Definition 4: Let p and q be propositions The exclusive or of p and q, denoted

$p \oplus q$, is the proposition that is true when one of p and q is true and is false otherwise.

Remark: The english language will use or as exclusive or at times. In this class we will always use or to mean the inclusive or.

Truth tables

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	T	T	F
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

Examples:

Let p be "Hiking is safe on the trail"

Let q be "Berries are ripe along the trail"

Express the following as an english sentence.

1) $p \wedge q$

"Hiking is safe on the trail and berries are ripe along the trail"

2) $p \vee q$

"Hiking is safe on the trail or berries are ripe along the trail"

3) $\neg p \oplus q$

"Hiking is safe on the trail or berries are ripe along the trail, but not both"

We can go from english sentences to propositional logic.

<u>Example</u>: Convert the following statement into propositional logic:

- My car is 5 years old and gray.

Denote p = "My car is 5 years old"

q = "My car is gray"

Then, the statement can be written as

$$\boxed{p \wedge q}$$

<u>Definition 5</u>:

Let p and q be propositions. The

<u>conditional statement</u> $p \to q$ is the proposition "If p then q". The conditional statement is false when p is true and q is false, and true otherwise.

In the conditional statement $p \to q$, p is the hypothesis and q is the conclusion.

Conditional statements can be expressed in the following ways:

"if p, then q"

"if p, q"

"p is sufficient for q"

"q if p"

"q when p"

"a necessary condition for p is q"

"q unless $\neg p$"

"p implies q"

"p only if q"

"a sufficient condition for q is p"

"q whenever p"

"q is necessary for p"

"q follows from p"

"q provided that p"

Example:

Let p be "Berries are ripe along the trail" and let q be "Grizzly bears have been seen in the area"

Express $p \to q$ as a statement in English.

"If berries are ripe along the trail then Grizzly bears have been seen in the area"

"Grizzly bears have been seen in the area only if berries are ripe along the trail"

Definitions:

Consider the conditional statement $p \to q$

The converse is $q \to p$

The inverse is $\neg p \to \neg q$

The contrapositive is $\neg q \to \neg p$.

Example:

Let p be "Berries are ripe along the trail" and let q be "Grizzly bears have been seen in the area"

Express the converse, inverse and contrapositive as a statement in English

Converse: $q \to p$.

"If Grizzly bears have been seen in the area then berries are ripe along the trail"

Inverse: $\neg p \to \neg q$

"If berries are not ripe along the trail then Grizzly bears have not been seen in the area."

Contrapositive: $\neg q \rightarrow \neg p$

If Grizzly bears have not been seen in the area, then berries are not ripe along the trail."

Definition 6: Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement is true when p and q have the same truth values and false otherwise.

Example

Let p be "Berries are ripe along the trail" and let q be "Grizzly bears have been seen in the area"

Express $p \leftrightarrow q$ as a statement in English.

Berries are ripe along the trail if and only if Grizzly bears have been seen in the area.

Truth Tables of conditional statements

p	q	$p \to q$	$p \leftrightarrow q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	T	T	T
T	F	F	F	T	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Remark: $p \to q$ and $\neg q \to \neg p$ have the same truth values depending on the truth values of p and q.

We say these two statements are logically equivalent.

We have now developed the basic operations to make compound propositions from simple propositions. We can use truth tables to determine the truth values of compound propositions.

Example:

Construct the truth table for $(p \vee \neg q) \to (p \wedge q)$. The ideas is that we want to break up this compound proposition into truth values of the operations that we have defined.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Remark: This table shows that $(p \vee \neg q) \to (p \wedge q)$ is logically equivalent to q. We will later show how you can derive this without a truth table.

Precedence of operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

So, $(p \vee \neg q) \rightarrow (p \wedge q)$ is the same thing as $p \vee \neg q \rightarrow p \wedge q$

However, it is best to use paranthesis to make statements clear!