Propositional logic cannot express the meaning of all statements in mathematics and natural language. For example, we know that

"Each of my cats has enough food"

No rules of propositional logic allow us to conclude

"Archie has enough food"

where Archie is one of my cats.

Similary we cannot use propositional logic to conclude from

"Gea is in the bedroom"

where Gea is my other cat, that

"One of my cats is in the bedroom".

To solve this we will introduce <u>predicate logic</u>.

Examples:

"$x > 3$"   Recall these were not
"$x = y + 3$"   propositions because the
"$x + y = z$"   truth value is determined
by the variables.

In the example "$x > 3$" we say the variable x is subject and the "is greater than" is the predicate. We can denote this as $P(x)$ where P denotes "is greater than 3" and x is the variable. $P(x)$ is said to be the value of the propositional function P at x. Once there is a value assigned to x, then $P(x)$ becomes a proposition and has a truth value.

Example:

Let $P(x)$ denote the statement

"$x>3$". Then $P(4)$ is the statement "$4>3$" which is true and

$P(3)$ is the statement "$3>3$" which is false.

Remark: We can have more than one variable. For example let $Q(x,y)$ be the statement "$x>y$".

We want to turn propositional functions into propositions. For example, intuitively statement "There is an integer x such that $x>3$" is a true statement even though "$x>3$" has no truth value. We turn proposition functions into propositions through a process called <u>quantification</u>.

<u>Definition 1:</u>

The <u>universal quantification</u> of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain"

We will use the notation

$$\forall x\, P(x) \quad \text{or} \quad \forall x \in D,\, P(x)$$

where D is the domain. We normally say "for all x in the domain, $P(x)$"

or for every

An element such that $P(x)$ is false is called a <u>counter example</u>.

Example:

Let $P(x)$ be the statement "$x+1>x$".

What is the truth value of $\forall x P(x)$?

Trick question, domain is not defined.

What is the truth value of $\forall x P(x)$

where the domain is all real numbers?

$P(x)$ is true for all x. So, $\forall x \in \mathbb{R}, P(x)$

is true.

Let $Q(x)$ be the statement "$x>3$"

What is the truth value of $\forall x \in \mathbb{R}, Q(x)$?

False, $Q(1)$ is false. $x=1$ is a

counter example.

What is the truth value of

$\forall x \in \mathbb{R}\,(x^2 \geq x)$ and $\forall x \in \mathbb{Z}\,(x^2 \geq x)$?

$(\frac{1}{2})^2 = \frac{1}{4} \not\geq \frac{1}{4}$. So $\forall x \in \mathbb{R}\,(x^2 \geq x)$ is

false. However, $x^2 \geq x$ is true

if and only if $x^2 - x = x(x-1) \geq 0$.

Consequently $x^2 \geq x$ if and only if

$(x \geq 0$ and $x \geq 1.)$ or $x \leq 0$ and $x \leq 1$.

So, $x^2 \geq x$ is true if and only if

$x \geq 1$ or $x \leq 0$. All integers satisfy

This so $\forall x \in \mathbb{Z}\,(x^2 \geq x)$ is true.

<u>Remark:</u> Truth values are dependent

on the domain.

Definition 2

The <u>existential quantification</u> of $P(x)$ is the proposition

"there exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$.

Example:

Let $P(x)$ be the statement "$x \geq 3$". What is the truth value of $\exists x \in \mathbb{Z} P(x)$?

$P(4)$ is true so $\exists x \in \mathbb{Z} P(x)$ is true.

Let $Q(x)$ be the statement "$x = x+1$". What is the truth value of $\exists x \in \mathbb{R} Q(x)$?

$Q(x)$ is false for every real number.

Hence, $\exists x \in \mathbb{R}\ Q(x)$ is false.

Remark: There are other quantifiers however the most used other quantifier is the uniqueness quantifier denoted by $\exists!$.

We say "There exists a unique x such that $P(x)$ is true". However, we can express uniqueness in terms of logical operators and universal/existential quantifiers.

Note that if the domain is finite then quantified statements can be expressed using propositional logic. In particular if $x_1, x_2, \ldots, x_n$ are the elements in the domain (where n is a positive integer) the $\forall x P(x)$ is the same as

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n).$$

Similarly, $\exists x P(x)$ is the same as

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee \cdots \vee P(x_n).$$

Sometimes we will want to restrict domains. For example only positive numbers. We can denote this as $\forall x>0\, P(x)$ and similarly for other restriction. What this means is $\forall x\,(x>0 \rightarrow P(x))$. We can do this with existential quantifiers as well. $\exists x \neq 0\, P(x)$ which is equivalent to $\exists x\,(x \neq 0 \wedge P(x))$.

<u>Example:</u>

What does $\forall x<0\,(x^2>0)$ mean where the domain is all real numbers?

"For every real number x less than 0,

x^2 is greater than 0" or

"For every real number x, if x is less then 0 then x^2 is greater than 0."

or

"The square of a negative number is positive."

<u>Remark</u>: The quantifiers have the highest precedence. So,

$\forall x P(x) \vee Q(x)$ is equivalent to

$(\forall x P(x)) \vee Q(x)$. BE CAREFUL WITH PARENTHESIS

Definition 3

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domains of discourse are used for the variables in the propositional functions. We use the notation $S \equiv T$ to indicate two statements S and T involving predicates and quantifiers are logically equivalent.

Example: Show that

$\exists x\,(P(x) \vee Q(x))$ is equivalent to

$(\exists x\, P(x)) \vee (\exists x\, Q(x))$.

Suppose $\exists x(P(x) \vee Q(x))$ is to true then there is some a in The domain that $P(a) \vee Q(a)$. If $P(a)$ is true then $\exists x P(x)$ is true if $Q(a)$ is true then $\exists x Q(x)$ is true. Hence we have $(\exists x P(x)) \vee (\exists x Q(x))$.

Now suppose $(\exists x P(x)) \vee (\exists x Q(x))$ is true. Then we have $\exists x P(x)$ or $\exists x Q(x)$ If $\exists x P(x)$ is true Then there is some a in The domain That $P(a)$ is true. Hence $P(a) \vee Q(a)$ is true. So, $\exists x(P(x) \vee Q(x))$. If $\exists x Q(x)$ is true then there is some a in The domain That $Q(a)$ is true. Hence $P(a) \vee Q(a)$ is true. So, $\exists x(P(x) \vee Q(x))$.

Sometimes we will want to negate quantified expressions. Recall that if the domain is finite that

$$\forall x P(x) \equiv P(x_1) \wedge \ldots \wedge P(x_n) \quad \text{and}$$

$$\exists x P(x) \equiv P(x_1) \vee \ldots \vee P(x_n)$$

so,

$$\neg(\forall x P(x)) \equiv \neg(P(x_1) \wedge \ldots \wedge P(x_n))$$

$$\equiv \neg P(x_1) \vee \ldots \vee \neg P(x_n)$$

$$\equiv (\exists x)(\neg P(x))$$

Demorgan's Law

So we should expect in general that

$$\neg(\forall x P(x)) \equiv (\exists x)(\neg P(x))$$

Suppose $\neg(\forall x P(x))$ is true. Then $\forall x P(x)$ is false. $\forall x P(x)$ is false if and only if there is an element x in the domain so that

$P(x)$ is false. This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true. Now finally there is an element for which $\neg P(x)$ is true if and only if $\exists x \neg P(x)$.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Note that $\neg$ switches the quantifiers.

Example: What are the negations of the statements $\forall x\,(x^2 > x)$ and $\exists x\,(x^2 = 2)$?

Using the rules $\neg(\forall x\,(x^2 > x)) \equiv \exists x\, \neg(x^2 > x)$

which is equivalent to $\exists x\,(x^2 \leq x)$.

$\neg(\exists x\,(x^2 = 2)) \equiv \forall x\, \neg(x^2 = 2)$

which is equivalent to $\forall x\,(x^2 \neq 2)$.