1.5 Nested Quantifiers

<u>Nested Quantifiers</u> are where one quantifier is within the scope of another, such as

$$\forall x \exists y (x+y=0).$$

This is the same as the statement $\forall x\, Q(x)$ where $Q(x)$ is the proposition $\exists y\, P(x,y)$ where $P(x,y)$ is "$x+y=0$".

<u>Example:</u>

Assume that the domain for x, y are all real numbers. The statement

$$\forall x \forall y\, (xy = yx)$$ say that $xy = yx$ for all real numbers x and y.

(This is the commutative law for multiplication)

Likewise the statement $\forall x \neq 0\, \exists y\; xy = 1$

say that for every real number $x \neq 0$ there is a real number y such that $xy = 1$.

(This states that every real number that is not zero has a multiplicative inverse"

$\forall x \forall y \forall z \, (x(yz) = (xy)z)$ is the associative law for multiplication of real numbers.

Example:

Translate into English the statement

$\forall x \forall y \, ((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

where the domain for x and y are all real numbers.

"For all real numbers x and y, if x is greater than 0 and y less than 0, then xy is less than 0."

Recall the first example,

$$\forall x \in \mathbb{R} \, \exists y \in \mathbb{R} (x+y=0)$$

This is the statement for every real number x there is a real number y such that $x+y=0$.

Does the order of the quantifiers make a difference? Is $\exists y \in \mathbb{R} \, \forall x \in \mathbb{R} (x+y=0)$ the same proposition?

No, this is equivalent to saying there exists a real number y such that for all real numbers $x+y=0$. This is not true since there is not a single number z such that $x+z=0$ for all x.

While $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (x+y=0)$ is true since we can take $y=-x$.

<u>Example</u>:

Are the statements $\forall x \forall y\, P(x,y)$ and $\forall y \forall x\, P(x,y)$ the same?

Yes, both statements are true if and only if $P(x,y)$ is true for all x and all y.

Are the statements $\exists x \exists y\, P(x,y)$ and $\exists y \exists x\, P(x,y)$ the same?

Yes, both statements are the same.

TABLE 1 Quantifications of Two Variables.		
Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Quantification of more than two variables is common.

Example:

let $Q(x,y,z)$ be the statement "$x+y=z$".

What are the truth values of $\forall x \forall y \exists z\, Q(x,y,z)$ and $\exists z \forall x \forall y\, Q(x,y,z)$ where the domain for x, y, and z are all real numbers?

$\forall x \forall y \exists z\, Q(x,y,z)$ is the statement
For all real numbers x and y there is a real number z such that $x+y=z$.

This is true since the sum of two real numbers is real.

$\exists z \forall x \forall y\, Q(x,y,z)$ is the statement
There is a real number z such that for all reals x and y, $x+y=z$.

This is false since not all real numbers sum to the same number.

<u>Example</u>

Translate the statement "The sum of two positive integers is always positive" into a logical expression.

$$\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}\, (x>0 \wedge y>0 \rightarrow x+y>0)$$

or $\forall x>0\, \forall y\geq 0\, (x+y>0)$.

Example: Translate "Every real number has an additive inverse" into a logical expression.

$\forall x \exists y\, (x+y=0)$.

Example: Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a.

$\lim_{x \to a} f(x) = L$ mean that $\forall \varepsilon > 0, \exists \delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

As a logical expression this is

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ (0 < |x-a| < \delta \to |f(x) - L| < \varepsilon)$.

Example:

Translate

$$\exists x \forall y \forall z ((F(x,y) \wedge F(x,z) \wedge (y \neq z) \rightarrow \neg F(y,z))$$

into English, where $F(a,b)$ means a and b are friends and the domain of x, y, z consists of all student in your school.

This says there is a student x such that for every student y and z in the school if x is friends with y and z and y and z are not the same student then y and z are not friend. In other words, there is a student none of whose friends are friends with each other.

Example

Express the statement "Everyone has exactly one best friend" as a logical expression.

Let $B(x,y)$ be the statement x and y are best friends. Let the domain for all variables be all people

$$\forall x \; \exists y \; (B(x,y) \wedge \forall z ((y \neq z) \rightarrow \neg B(x,z)))$$

Example:

Express the negation of $\forall x \exists y (xy=1)$ so that no negation preceds a quantifier.

$$\neg(\forall x \exists y (xy=1)) \equiv \exists x \neg (\exists y \; (xy=1))$$

$$\equiv \exists x \; \forall y \; \neg (xy=1)$$

$$= \exists x \; \forall y \; (xy \neq 1).$$

Example:

Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world".

Let $P(w,f)$ be "w has taken f"

and let $Q(f,a)$ be "f is a flight on a".

The domain of w is all women

The domain of f is all flights

The domain of a is all airlines.

$$\neg(\exists w\,\forall a\,\exists f\,(P(w,f)\wedge Q(f,a))$$

$$\equiv \forall w\,\neg(\forall a\,\exists f\,(P(w,f)\wedge Q(f,a))$$

$$\equiv \forall w\,\exists a\,\neg(\exists f\,(P(w,f)\wedge Q(f,a))$$

$$\equiv \forall w\,\exists a\,\forall f\,\neg(P(w,f)\wedge Q(f,a))$$

$$\equiv \forall w\,\exists a\,\forall f\,(\neg P(w,f)\vee\neg Q(f,a)).$$

Example:

Use quantifiers and predicates to express the fact that $\lim_{x \to a} f(x)$ does not exist.

$$\neg\left(\forall \varepsilon>0\, \exists \delta>0\, (0<|x-a|<\delta \to |f(x)-L|<\varepsilon)\right)$$

$$\equiv \exists \varepsilon>0\, \neg\left(\exists \delta>0\, (0<|x-a|<\delta \to |f(x)-L|<\varepsilon)\right)$$

$$\equiv \exists \varepsilon>0\, \forall \delta>0\, \neg(0<|x-a|<\delta \to |f(x)-L|<\varepsilon)$$

$$\equiv \exists \varepsilon>0\, \forall \delta>0\, (0<|x-a|<\delta \wedge |f(x)-L|\geq\varepsilon).$$

Recall $p \to q \equiv \neg p \vee q$

So $\neg(p \to q) \equiv \neg(\neg p \vee q)$

$\equiv p \wedge \neg q$