Sequences

Examples

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right\}$$

Examples

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\left\{\frac{(-1)^n}{2^n}\right\}$$

$\left\{ -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots \right\}$

Examples

$$a_n = \begin{cases} 0, & n \text{ odd}, \\ n, & n \text{ even}. \end{cases}$$

$\{0, 2, 0, 4, \ldots\}$

$$\left\{ \frac{(-1)^n n}{2n-1} \right\}$$

$\left\{ \frac{-1}{2-1}, \frac{2}{4-1}, \frac{-3}{6-1}, \frac{4}{8-1}, \frac{-5}{10-1}, \ldots \right\}$

$\{-1, 2/3, -3/5, 4/7, -5/9, 6/11, -7/13, \ldots\}$

DEFINITION Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases—that is, if a_n can be made arbitrarily close to L by taking n sufficiently large—then we say $\lim_{n \to \infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

Intuitive Definition

Let $\{a_n\}$ be a sequence. $a_n \to L$ if a_n gets as close as we want as n gets bigger.

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$$|a_n - 1| = \left|\frac{n-2}{n} - 1\right| = \left|\frac{n-2}{n} - \frac{n}{n}\right| = \frac{2}{n}$$

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and $2/n \to 0$.
Technique: Identify L. Show that $(a_n - L) \to 0$.

Converges

$$\left\{ \frac{n}{3n+4} \right\}$$

Show converges to 1/3:

$$\left\{ \frac{1}{3+4}, \frac{2}{6+4}, \frac{3}{9+4}, \frac{4}{16}, \ldots \right\}$$

$$\left\{ \frac{1}{7}, 2/10, 3/13, 4/16, 5/19, 6/22, \ldots \right\}$$

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$$= \left| \frac{3n - (3n+4)}{3(3n+4)} \right| = \left| \frac{-4}{3(3n+4)} \right| = \frac{4}{(3(3n+4))}$$

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factor out something you know:

$$\frac{4}{3(3n+4)} = \left(\frac{1}{n}\right)\left(\frac{4}{3(3+4/n)}\right).$$

We know $1/n \to 0$, and the second term is bounded, so

$$\left| \frac{-4}{3(3n+4)} \right| \to 0$$

Divergence

Divergent just means not convergent. Example:

$$a_n = (-1)^n \left(\frac{n-1}{n} \right).$$

For odd $n = 2k + 1$,

$$a_n = (-1)^{2k+1} \frac{(2k+1)-1}{2k+1} = -\frac{2k}{2k+1} \to -1.$$

Divergent

$$\{(-1)^n\}$$

$\{-1, 1, -1, 1, -1, 1\}$

This is like $\sin\left(\frac{n\pi}{2} + \pi\right)$ where I discussed how $\lim_{x \to \infty} \sin(x)$ does not exist.

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Technique: show two parts of sequence converge to two different things. That means the sequence diverges.

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Either $L \geq 0$, or $L < 0$.

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Either $L \geq 0$, or $L < 0$.

If $L \geq 0$, then $1 + L \geq 1$, so $|1 + L| \geq 1$, which means that the sequence stays away from L for every odd n. Similar argument for $L < 0$ and even n.

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Exercise

Show that the sequence $\{5n\}$ diverges.

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Show that the sequence $\{5n\}$ diverges. Let L be a real number. If N is large enough, then $5N \geq L + 1$. Then for every $n \geq N$,

$$5n - L \geq 5N - L \geq 1$$

so $5n$ stays away from L. $\rightsquigarrow$ divergent.

Exercise

Show that the sequence

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THEOREM 10.1 Limits of Sequences from Limits of Functions
Suppose f is a function such that $f(n) = a_n$, for positive integers n. If $\lim_{x \to \infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L, where L may be $\pm\infty$.

__Remark:__ If $\lim_{x \to \infty} f(x)$ DNE the sequence can still converge.

__For example:__ $f(n) = \sin(2\pi n)$ gives the sequence $\{0, 0, 0, \ldots\}$ but $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin(x)$ DNE.

__Example:__ $\left\{\frac{2n-1}{4n-3}\right\}$ converges to $1/2$

$f(n) = \frac{2n-1}{4n-3}$ gives the sequence.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x-1}{4x-3} = \frac{\infty}{\infty}$ so use L'Hopital's rule

$= \lim_{x\to\infty} \frac{2}{4} = \frac{1}{2}.$

Algebraic limit theorem

Suppose $\{a_n\}$ converges to L_1 and $\{b_n\}$ converges to L_2. Then
1) $a_n + b_n \to L_1 + L_2$
2) If c is a real number, then $ca_n \to cL_1$
3) $a_n b_n \to L_1 L_2$
4) If $b_n \neq 0$ for every n and $L_2 \neq 0$, then $\frac{a_n}{b_n} \to \frac{L_1}{L_2}$.

Squeeze Theorem

Let $\{a_n\}$ converge to L, and $\{b_n\}$ converge to L. Suppose $\{c_n\}$ is a sequence with $a_n \leq c_n \leq b_n$ for every n. Then $c_n \to L$.

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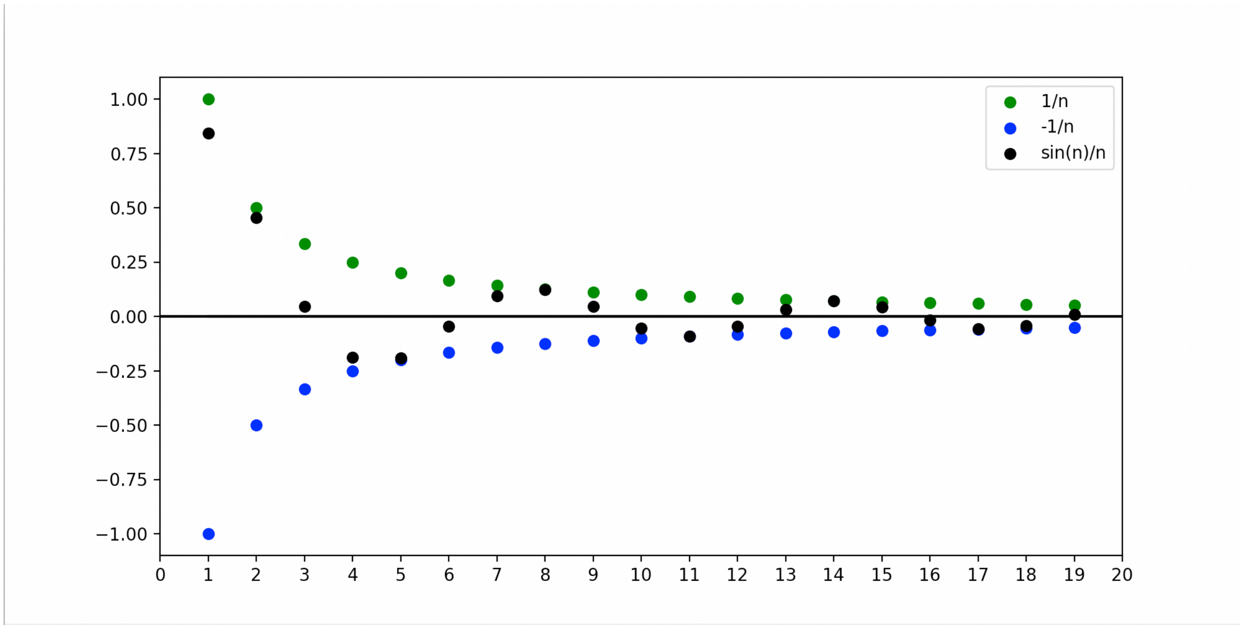
Example:

$$a_n = -1/n, \; b_n = 1/n, \; c_n = \sin(n)/n.$$



Notice: $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

and $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$

Therefore $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0.$

Formaly $-1 \leq \sin(n) \leq 1$

so $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

Hence we can show this holds without the graph. The graph gives an idea but it is not a proof.

DEFINITIONS Terminology for Sequences

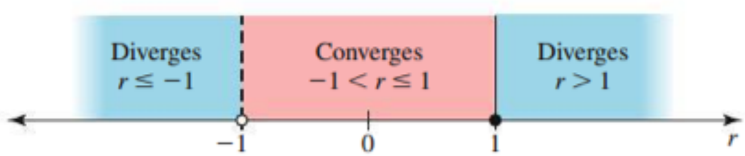
- $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, \ldots\}$.
- $\{a_n\}$ is **nondecreasing** if $a_{n+1} \geq a_n$; for example, $\{1, 1, 2, 2, 3, 3, \ldots\}$.
- $\{a_n\}$ is **decreasing** if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -1, \ldots\}$.
- $\{a_n\}$ is **nonincreasing** if $a_{n+1} \leq a_n$; for example, $\{0, -1, -1, -2, -2, \ldots\}$.
- $\{a_n\}$ is **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction).
- $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$, for all relevant values of n, and $\{a_n\}$ is **bounded below** if there is a number N such that $a_n \geq N$, for all relevant values of n.
- If $\{a_n\}$ is bounded above and bounded below, then we say that $\{a_n\}$ is a **bounded** sequence.

THEOREM 10.3 Geometric Sequences

Let r be a real number. Then

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.



Geometric sequences appear a lot in nature. Bacteria Growth / compound interest.

THEOREM 10.5 Bounded Monotonic Sequence

A bounded monotonic sequence converges.

Monotone examples

Example:

$$\{1 - 1/n\}$$

is monotone increasing and bounded by 1.

Monotone examples

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$$\{1 - 1/n\}$$

is monotone increasing and bounded by 1.
Example: recursive sequence
Let $a_1 = \sqrt{3}$, and for $n \geq 1$,

$$a_{n+1} = \sqrt{3 + a_n}.$$

Technique: prove monotone and bounded, then solve for limit.

$a_1 = \sqrt{3}$ $\quad a_2 = \sqrt{3+\sqrt{3}} > \sqrt{3}$ $\quad a_3 = \sqrt{3 + \_\_\_} > a_2$

$a_4 = \sqrt{3+a_3} > \sqrt{3+a_2}$ since $a_3 \geq a_2$

proof: Assume $a_n > a_{n-1}$ for $n \geq 2$.

Then $a_{n+1} = \sqrt{3+a_n} \geq \sqrt{3+a_{n-1}} = a_n$

by the assumption.

Bounded: Suppose $a_n < 97$ Then

$a_{n+1} = \sqrt{3+a_n} \leq \sqrt{3+97} \leq 10$.

We showed that $a_1 < 97$ and $a_{n+1} \leq 10 < 97$

hence this sequence is bounded.

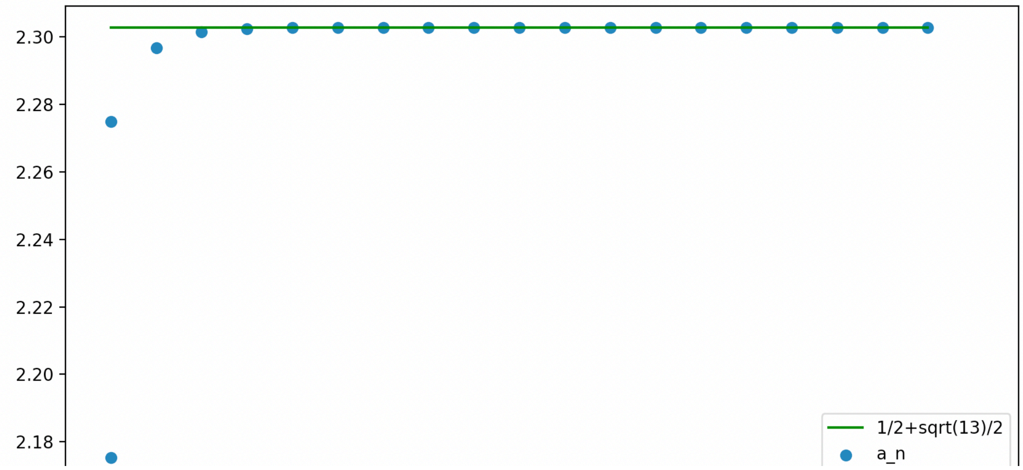
So MCT this sequence converges.

To what? Solve $x = \sqrt{3+x}$

$x^2 = 3+x \qquad x^2 - x - 3 = 0$

$$\frac{1 \pm \sqrt{1+12}}{2} = \boxed{\frac{1}{2} + \frac{\sqrt{13}}{2}}$$

We can see from the graph this is the limit.



THEOREM 10.6 Growth Rates of Sequences
The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers p, q, r, s, and $b > 1$.

This idea is very important for series and comparison tests. It also comes up in algorithms in computer science.

DEFINITION Limit of a Sequence

The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if, given any $\varepsilon > 0$, it is possible to find a positive integer N (depending only on ε) such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

If the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n \to \infty} a_n = L.$$

A sequence that does not converge is said to **diverge**.

Does the following sequence converge or diverge?

(A) $\left\{ \dfrac{n^{1000}}{2^n} \right\}$

(B) $\left\{ \dfrac{\ln(n^2)}{\ln(3n)} \right\}$

(C) $\left\{ \dfrac{\cos(n\pi/2)}{\sqrt{n}} \right\}$

(D) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ $a_0 = 2$

(E) $\{ 1 + \cos(n) \}$

(A) Theorem 10.6 gives $\left\{\frac{n^{1000}}{2^n}\right\} \to 0$

converges

(B) $\lim_{n\to\infty} \frac{\ln(n^2)}{\ln(3n)} = \lim_{n\to\infty} \frac{2\ln(n)}{\ln(3n)} = \lim_{n\to\infty} \frac{2\,1/n}{1/n}$

$= \boxed{2}$

converges

(C) $\cos(n\pi/2) = 0$ for all n

$\left\{\frac{\cos(n\pi/2)}{\sqrt{n}}\right\} = \left\{\frac{0}{\sqrt{1}}, \frac{0}{\sqrt{2}}, \frac{0}{\sqrt{3}}, \ldots\right\}$

$= \{0, 0, 0, 0, \ldots\}$

converges to $\boxed{0.}$

(D) $a_0 = 2 \quad a_1 = \frac{1}{2}\left(2 + \frac{2}{3}\right) = \frac{8}{6} = \frac{4}{3}$

$a_2 = \frac{1}{2}\left(\frac{4}{3} + \frac{6}{4}\right) = \frac{1}{2}\left(\frac{16+18}{12}\right)$

$= \frac{34}{24} = \frac{17}{12} < \frac{4}{3}$

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

So if a limit exists we know that $L = \sqrt{2}$

$$2L = \left(L + \frac{2}{L}\right)$$

$$2L^2 = L^2 + 2$$

$$L^2 = 2$$

$$\boxed{L = \sqrt{2}}$$

$a_n > \sqrt{2}$ Then

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \frac{1}{2}\left(a_n + \sqrt{2}\right) \geq \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

So, $a_n \geq 1$ for all n.

Suppose $\sqrt{2} < a_n$ Then

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) < \frac{1}{2}\left(a_n + \sqrt{2}\right)$$

$$\leq \frac{1}{2}(2a_n)$$

$$= a_n.$$

So this is monotonically decreasing. and bounded. Hence $a_n \to \sqrt{2}$

(E) This diverges due to the $\cos(n)$ term. For every $N > 0$ there are points $n_1, n_2 > N$ such that $\cos(n_1) > \sqrt{2}/2$ and $\cos(n_2) < -\frac{\sqrt{2}}{2}$.