Recursive sequences and introduction to infinite series

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Generalize: 1) Check if P_1 holds. 2) Now show for any k, $P_k \implies P_{k+1}$. Then $P_1 \implies P_2 \implies P_3 \implies P_4 \dots$. "Today's problem P_n follows from yesterday's problem P_{n-1}, which follows from two days ago P_{n-2}, all the way back to P_1.

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Does $P_k \implies P_{k+1}$? Assume P_k:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Then verify P_{k+1}:

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \ldots$$

Recursive example

Example: recursive sequence
Let $a_1 = \sqrt{3}$, and for $n \geq 1$,

$$a_{n+1} = \sqrt{3 + a_n}.$$

Technique: prove monotone and bounded, then solve for limit.
IF the limit exists and $a_n \to L$, take $\lim_{n\to\infty}$ of both sides
$L = \sqrt{3 + L}$, so

$$L = \frac{1 \pm \sqrt{13}}{2}.$$

Have to prove monotone and bounded first!!!

Other recursive sequence: $a_1 = 10$, and $a_{n+1} = a_n^2 - 2$. Apply limit **without showing in converges first**, $L = L^2 - 2$, so $L = 2$ or -1. WRONG!

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$$2^{-1}s = 1/2 + 1/4 + 1/8 + ...$$

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$$s - 2^{-1}s = 1, \text{ or } s = \frac{1}{1 - 1/2}.$$

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Assumed the series converged before manipulating it.

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Let r be a real number. The infinite series $\sum_{j=0}^{\infty} r^j$ converges if and only if $|r| < 1$. In this case, it converges to $1/(1-r)$.

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Have to use partial sums!!!

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Each s_N is a number, so $\{s_N\}_{N=1}^{\infty}$ is a sequence of numbers. If it converges to s, we say $\sum_{n=1}^{\infty} a_n$ converges to s.

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$s_N \to 1/2$, so series converges to $1/2$

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converges to $\frac{8/5}{3/5} = 8/3$.

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$$= \sum_{n=1}^{\infty} \frac{2^7 2^n}{3^{-4} 3^n}$$

$$= \frac{2^7}{3^{-4}} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

$$= \frac{2^7}{3^{-4}} \left(\frac{2/3}{1 - 2/3}\right).$$

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Monotone convergence theorem.

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$$\begin{aligned} s_N = {} & (2 - 2/3) + (2/2 - 2/4) + (2/3 - 2/5) + \ldots \\ & + (2/(N-1) - 2/(N+1)) + (2/N - 2/(N+2)) \end{aligned}$$

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$s_N \to 3$

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That is a geometric series sum, so $s_N \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.
Bounded!

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A divergence theorem

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$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges!

Harmonic “series”

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In general, $s_{2^N} \geq 1 + 1/2 + N(1/2)$ so it diverges since $\{s_N\}$ is unbounded.