

Integral test

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Harmonic “series”

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$$1 + 1/2 + 1/3 + \dots$$

$$= 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8)$$

$$+ (1/9 + \dots + 1/16) + \dots$$

$$\geq 1 + 1/2 + 2(1/4) + 4(1/8) + 8(1/16) + \dots$$

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$$\begin{aligned} &1 + 1/2 + 1/3 + \dots \\ &= 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) \\ &\quad + (1/9 + \dots + 1/16) + \dots \\ &\geq 1 + 1/2 + 2(1/4) + 4(1/8) + 8(1/16) + \dots \end{aligned}$$

In general, $s_{2N} \geq 1 + 1/2 + N(1/2)$ so it diverges since $\{s_N\}$ is unbounded.

Integral test

Theorem: Let $f(x)$ be a function on $[1, \infty)$ which is

1) positive

2) decreasing.

For each n , let $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if

$$\int_1^{\infty} f(x) dx$$

converges.

Why does this work?

↓

Integral test

This means you can use your intuition about improper integrals:

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In fact, we can use the integral to estimate the sum if it converges:

$$\int_1^{N+1} f(x) dx \leq s_N \leq a_1 + \int_1^N f(x) dx.$$

Example

Determine whether the sequence

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

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$$s_N = a_1 + a_2 + \dots + a_N \leq 1/2 + \int_1^N \frac{1}{1+x^2} dx$$

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$$s_N = a_1 + a_2 + \dots + a_N \leq 1/2 + \int_1^N \frac{1}{1+x^2} dx$$

$$\searrow s_N \geq \int_1^{N+1} \frac{1}{1+x^2} dx.$$

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$$\frac{1}{3} \leq \sum_{n=1}^{\infty} \frac{1}{3n^2} \leq \frac{2}{3}.$$

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Determine if the following series converges or diverges and estimate the sum if it converges.

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$$s_N \geq \int_1^{N+1} \frac{3}{\sqrt{x+2}} dx = 6(\sqrt{N+3} - \sqrt{3})$$

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$$\int_3^{N+1} \frac{1}{4(x-2)^3} dx \leq \frac{1}{4} + \int_3^N \frac{1}{4(x-2)^3} dx$$

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$$\int_3^{N+1} \frac{1}{4(x-2)^3} dx \leq \frac{1}{4} + \int_3^N \frac{1}{4(x-2)^3} dx$$

or

$$\frac{1}{8} - \frac{1}{8(N-1)^2} \leq s_N \leq \frac{3}{8} - \frac{1}{8(N-2)^2}$$

Remainder estimate

In fact, we can use the integral to estimate the sum if it converges:

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Can use integral to estimate *remainder*. Under the assumptions of the theorem, if $\sum a_n$ converges to s , the remainder $R_N = s - s_N$:

$$R_N = a_{N+1} + a_{N+2} + \dots$$

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Shift integral:

$$a_{N+1} + a_{N+2} + \dots \leq \int_N^{\infty} f(x) dx,$$

$$a_{N+1} + a_{N+2} + \dots \geq \int_{N+1}^{\infty} f(x) dx.$$

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$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$$

converges. How many terms do you need so that s_N is within $1/1000$ of s ?

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$$\int \frac{1}{(3x-1)^4} dx = -\frac{1}{9}(3x-1)^{-3} + C.$$

$$\frac{1}{9}(3(N+1)-1)^{-3} \leq R_N \leq \frac{1}{9}(3N-1)^{-3}$$

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$$\frac{1}{9}(3(N+1)-1)^{-3} \leq R_N \leq \frac{1}{9}(3N-1)^{-3}$$

$$\frac{1}{9}(3N-1)^{-3} = \frac{1}{1000} \rightsquigarrow N = \frac{1}{3} + \frac{10}{3 \cdot 9^{1/3}}$$

Any bigger N works.

Exercise - p test

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

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For which values of p does the following series converge?

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What about $p = 1$?

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$$\int_N^{\infty} xe^{-x^2} dx = \frac{1}{2} e^{-N^2} = 10^{-3}$$

$$\rightsquigarrow N \geq \sqrt{\ln(10^3/2)}$$

Comparison test

Theorem: Let $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ be infinite series with $a_n \geq 0$ and $b_n \geq 0$ for every n .

1) If $\sum_1^\infty b_n$ converges and $b_n \geq a_n$ for every n then $\sum_1^\infty a_n$ converges

2) If $\sum_1^\infty b_n$ diverges and $b_n \leq a_n$ for every n then $\sum_1^\infty a_n$ diverges

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$$\frac{1}{3^n + 1} \leq \frac{1}{3^n}.$$

$\sum_1^{\infty} \frac{1}{3^n}$ converges, so does $\sum_1^{\infty} \frac{1}{3^n + 1}$.

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Show that the series

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$$n^2 + 2n + 2 = (n^2 + 2n + 1) + 1 = (n + 2)^2 + 1 \geq n^2$$

so

$$\frac{1}{n^2 + 2n + 2} \leq \frac{1}{n^2}$$

p test $\rightsquigarrow \sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$ converges.

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$$\sqrt{n^2 - 1} \leq \frac{1}{b_n}.$$

$$(\sqrt{n^2 - 1})^2 = n^2 - 1 \leq n^2,$$

so

$$\frac{1}{\sqrt{n^2 - 1}} \geq \frac{1}{n}$$

Example - more subtle estimate

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$n \geq 2$, so $\frac{1}{2}n^2 \geq 2 \rightsquigarrow$

$$n^2 - 1 = \frac{1}{2}n^2 + \frac{1}{2}n^2 - 1 \geq \frac{1}{2}n^2 + 2 - 1 \geq \frac{1}{2}n^2,$$

so

$$\frac{1}{n^2 - 1} \leq \frac{2}{n^2}.$$

p test.