Comparison test

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$\int_1^\infty f(x)\,dx$ converges and $f(x) \geq g(x) \geq 0$
Then $\int_1^\infty g(x)\,dx$ converges

Theorem: Let $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ be infinite series with $a_n \geq 0$ and $b_n \geq 0$ for every n.

1. If $\sum_1^\infty b_n$ converges and $b_n \geq a_n$ for every n then $\sum_1^\infty a_n$ converges
2. If $\sum_1^\infty b_n$ diverges and $b_n \leq a_n$ for every n then $\sum_1^\infty a_n$ diverges

$\int_1^\infty f(x)\,dx$ diverges and $g(x) \geq f(x) \geq 0$

then $\int_1^\infty g(x)\,dx$ diverges

Intuition

Note: Need everything to have **positive terms**!

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If $\sum b_n$ converges and $a_n \leq b_n$, then b_n pushes a_n down and makes the sum smaller.

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Ask: what does a_n look like that I already know?

If $\sum b_n$ converges and $a_n \leq b_n$, then b_n pushes a_n down and makes the sum smaller.

If $\sum b_n$ diverges and $a_n \geq b_n$, then b_n pushes a_n up and makes the sum even larger.

Example

Show that

$$\sum_{1}^{\infty} \frac{1}{3^n + 1} \approx \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

converges.

$$= \frac{1}{3}\left(\frac{1}{1-1/3}\right)$$

$$= \frac{1}{3}\left(\frac{1}{2/3}\right) = \frac{1}{2}$$

$$\frac{1}{3^n+1} \leq \frac{1}{3^n} ?$$

Since $3^n+1 \geq 3^n$ we get

$$\frac{1}{3^n} \geq \frac{1}{3^n+1}$$

So we have that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+1} \leq \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$ so the sum converges

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I need $3^n + 1 \geq \frac{1}{b_n}$.

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$$\frac{1}{3^n + 1} \leq \frac{1}{3^n}.$$

$\sum_1^\infty \frac{1}{3^n}$ converges, so does $\sum_1^\infty \frac{1}{3^n+1}$.

Exercise

Does the following series converge or diverge?

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Converges p-series test

converges. Use comparison test.

$n^2 + 2n + 2 \geq n^2$ for $n \geq 1$

$$\frac{1}{n^2} \geq \frac{1}{n^2+2n+2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Converges

so we know that $\sum_{n=1} \frac{1}{n^2+2n+2}$ converges.

Exercise

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 2}$$

converges. Use comparison test.
Looks like $1/n^2$:

Exercise

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\underline{n^2+2n+2}} \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \to \text{converg}$$

converges. Use comparison test.
Looks like $1/n^2$:

$$n^2 + 2n + 2 \geq n^2$$

so

$$\frac{1}{\underline{n^2+2n+2}} \leq \frac{1}{\underline{n^2}}$$

p test $\rightsquigarrow \sum_1^\infty \frac{1}{n^2+2n+2}$ converges.

Divergence comparison

Show that the following series diverges using comparison test:

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}} \quad \sum \frac{1}{(n^2)^{1/2}} = \sum \frac{1}{n}$$

$$n^2 - 1 \leq n^2$$

$$\sqrt{n^2-1} \leq \sqrt{n^2} = n$$

$$\frac{1}{n} \leq \frac{1}{\sqrt{n^2-1}}$$

So

$$\boxed{\infty = \sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-1}}} \to \infty$$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-1}}$ diverges as well.

Divergence comparison

Show that the following series diverges using comparison test:

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}}$$

Looks like $1/n$. **Important:** If I want to show

$$\frac{1}{\sqrt{n^2 - 1}} \geq b_n,$$

need

$$\sqrt{n^2 - 1} \leq \frac{1}{b_n}.$$

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need

$$\sqrt{n^2 - 1} \leq \frac{1}{b_n}.$$

$$(\sqrt{n^2 - 1})^2 = n^2 - 1 \leq n^2,$$

so

$$\frac{1}{\sqrt{n^2 - 1}} \geq \frac{1}{n}$$

Example - more subtle estimate

Does the following series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \sim \sum \frac{1}{n^2}$$

$$n^2 - 1 \le n^2$$

$$\frac{1}{n^2} \le \frac{1}{n^2-1}$$

$$C = \sum_{n=2}^{\infty} \frac{1}{n^2} \le \sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

$\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converges because of the

integral $\int_2^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} - \text{Arctan}(2)$

$n^2-1 \le n^2+1$

$\frac{1}{n^2+1} \le \frac{1}{n^2-1}$ $\qquad \frac{1}{n^2+1}$ Doesn't work

$n^{3/2} \le n^2-1$?

$n^{3/2} \le n^2-1$ hold

$$\sum_{n=1}^{\infty} \frac{1}{n^2-1} \le \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = C < \infty$$

so we have $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ so $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

Compare with $\frac{2}{n^2}$ instead of $\frac{1}{n^2}$

$$n^2-1 = \frac{1}{2}n^2 + \frac{1}{2}n^2 - 1 \ge \frac{1}{2}n^2 + 2 - 1$$

since $n \ge 2$

$$\geq \frac{1}{2} n^2 + 1$$

$$\geq \frac{n^2}{2}$$

So we get

$$\boxed{\frac{1}{n^2-1} \leq \frac{10}{n^2}}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \leq \sum_{n=2}^{\infty} \frac{2}{n^2} = C < \infty.$$

$$\boxed{\frac{1}{n^2-1} \leq \frac{1}{n^{3/2}}}$$

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Show the following sum converges:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Looks like $1/n^2$.

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$$n^2 - 1 \leq n^2$$

inequality goes the wrong way!

Example - more subtle estimate

Show the following sum converges:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Looks like $1/n^2$.

$$n^2 - 1 \leq n^2$$

inequality goes the wrong way!
$n \geq 2$, so $\frac{1}{2}n^2 \geq 2 \rightsquigarrow$

$$n^2 - 1 = \frac{1}{2}n^2 + \frac{1}{2}n^2 - 1 \geq \frac{1}{2}n^2 + 2 - 1 \geq \frac{1}{2}n^2,$$

so

$$\frac{1}{n^2 - 1} \leq \frac{2}{n^2}.$$

p test.

Limit comparison test

Last example worked because

$$\frac{n^2-1}{n^2} \to 1,$$

meaning they look the same as $n \to \infty$.

$\rightarrow b_n$

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^2-1}\right] \; a_n \quad \text{and} \quad \sum_{n=1}^{\infty}\left[\frac{1}{n^{3/2}}\right]$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^2-1}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2}}{n^2-1} = 0$$

Limit comparison test

Last example worked because

$$\frac{n^2-1}{n^2} \to 1,$$

meaning they look the same as $n \to \infty$.

Theorem: Let $\sum a_n$ and $\sum b_n$ are two series with *positive* terms. Then:

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} \to \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Intuition

Need positive again!

Intuition

Need positive again!
1) says a_n and b_n are really the same at ∞, so behaviour of one is the same as the behaviour of the other.
2) says if b_n is much bigger than a_n as $n \to \infty$, then b_n pushes a_n down to make the sum smaller.
3) says if b_n is much smaller than a_n as $n \to \infty$, then b_n pushes a_n up to make the sum ~~smaller~~.
Larger

$\lim_{k\to\infty} \frac{a_k}{b_k} = L$. Then we know that Given $\varepsilon > 0$ $\exists N$ sufficiently large so that if $k > N$ we have $\left|\frac{a_k}{b_k} - L\right| < \varepsilon$.

$$\left|\frac{a_k}{b_k} - L\right| < \frac{L}{2}$$ which implies

$$-\frac{L}{2} < \frac{a_k}{b_k} - L < \frac{L}{2}$$

$$\frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2}$$

$$\underline{\frac{L}{2}} \cdot b_k < a_k < \underline{\frac{3L}{2}} \cdot b_k$$

$$\sum_{k=1}^{\infty} a_k = \underbrace{\sum_{k=1}^{N} a_k}_{} + \underbrace{\sum_{k=N+1}^{\infty} a_k}_{}$$

finite

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{N} b_k + \sum_{k=N+1}^{\infty} b_k$$

$$\frac{L}{2}\sum_{k=N+1}^{\infty} b_k \le \sum_{k=N+1}^{\infty} a_k \le \frac{3L}{2}\sum_{k=N+1}^{\infty} b_k$$

If $\sum_{k=1}^{\infty} b_n$ converges we know $\frac{3L}{2}\sum_{k=N+1}^{\infty} b_k$ converge

So by comparison test $\sum_{k=N+1}^{\infty} a_k$ converges.

Example

$$\sum_{n=2}^{\infty} \frac{2n+3}{\sqrt{n^3-1}} \approx \frac{n}{(n^3)^{1/2}} = \frac{n}{n^{3/2}}$$

$$= \frac{1}{n^{1/2}}$$

Example

a_n

$$\sum_{n=2}^{\infty} \left(\frac{2n+3}{\sqrt{n^3-1}}\right) \qquad \left(\frac{1}{n^{1/2}}\right) b_n$$

looks like $n/n^{3/2} = 1/n^{1/2}$.

$$\lim_{n\to\infty} \frac{\left(\frac{2n+3}{\sqrt{n^3-1}}\right)}{\left(1/n^{1/2}\right)} = \lim_{n\to\infty} n^{1/2}\left[\frac{(2n+3)}{\sqrt{n^3-1}}\right]$$

$$= \lim_{n\to\infty} \frac{\cancel{n^{3/2}}\left(2+3/n\right)}{\cancel{n^{3/2}}\sqrt{1-1/n^3}} = 2 > 0$$

$3/n \to 0$, $1/n^3 \to 0$

Example

$$\sum_{n=2}^{\infty} \frac{2n+3}{\sqrt{n^3 - 1}}$$

looks like $n/n^{3/2} = 1/n^{1/2}$.

$$\lim_{n\to\infty} \frac{\frac{2n+3}{\sqrt{n^3-1}}}{1/n^{1/2}} = \lim_{n\to\infty} \frac{n^{3/2}(2+3/n)}{n^{3/2}(\sqrt{1-1/n^3})} = 2 > 0.$$

Example

$$\sum_{n=2}^{\infty} \frac{2n+3}{\sqrt{n^3-1}}$$

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$$\lim_{n\to\infty} \frac{\frac{2n+3}{\sqrt{n^3-1}}}{1/n^{1/2}} = \lim_{n\to\infty} \frac{n^{3/2}(2+3/n)}{n^{3/2}(\sqrt{1-1/n^3})} = 2 > 0.$$

$\sum \frac{1}{n^{1/2}}$ diverges, so does our sum.

Example and Exercise

Determine if the sums converge:

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+1)^{1/3}} \approx \frac{1}{(n^3)^{1/3}} = \frac{1}{n}$$

Diverges ~~but~~ but

$$\frac{1}{(n^3+1)^{1/3}} \le \frac{1}{n} \quad \text{so a}$$

straight comparison doesn't work.

$$\lim_{n\to\infty} \frac{1/(n^3+1)^{1/3}}{1/n} = \lim_{n\to\infty} \frac{n}{(n^3+1)^{1/3}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\cancel{1/n^3}^{\,0})^{1/3}}$$

$$= 1.$$

Since $\sum_{n=1}^{\infty} 1/n$ diverges we know that $\sum_{n=1}^{\infty} \frac{1}{(n^3+1)^{1/3}}$ diverges as well.

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+1)^{1/3}} \geq \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$\frac{1}{2n} \leq \frac{1}{(n^3+1)^{1/3}}$$

$$\sum \frac{1}{(n^3+1)^{1/2}} \leq \sum \frac{1}{n^{3/2}} \quad \text{converges}$$

Example and Exercise

Determine if the sums converge:

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+1)^{1/3}}$$

diverges

Example and Exercise

Determine if the sums converge:

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+1)^{1/3}}$$

diverges

$$\sum_{n=1}^{\infty} n3^{-n} = \sum_{n=1}^{\infty} \frac{n}{3^n} \sim \frac{1}{3^n}$$

$$\frac{n}{3^n} \geq \frac{1}{3^n}$$

This comparison does not work

$$\lim_{n\to\infty} \frac{n/3^n}{1/3^n} = \lim_{n\to\infty} n = \underline{\infty}$$

Only ~~works~~ works if we want $\sum a_n$ diverges because $\sum b_n$ diverges.

$$\underline{n \le 2^n}$$

$$\sum_{n=1}^{\infty} \frac{n}{3^n} \le \sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

converges.

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \le \sum_{n=1}^{\infty} \frac{(1.5)^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

$$\underline{n^2 \le 1.5^n}$$

Example and Exercise

Determine if the sums converge:

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$$\sum_{n=1}^{\infty} n3^{-n}$$

converges

$$\sum_{n=1}^{\infty} \sin(1/n)$$

$$\lim_{n\to\infty} \sin(1/n) = \sin(0) = 0$$

So the divergence test is inconclusive.

$f(x) = \sin(x) \approx x$ near $x=0$.

$f'(x) = \cos(x) = \underline{1}$ at $x=0$

$f''(x) = -\sin(x) = \underline{0}$ at $x=0$

$g(x) = x$

$g'(x) = 1$ at $x=0$

$\underline{g''(x) = 0}$ at $x=0$

This tells us $\sin(\frac{1}{n}) \approx \frac{1}{n}$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \approx \sum_{n=1}^{\infty} \frac{1}{n}.$$

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{1/n} = \lim_{n\to\infty} \frac{\frac{-1}{n^2}\cos(1/n)}{-1/n^2}$$

$$= \lim_{n\to\infty} \cos(1/n)$$

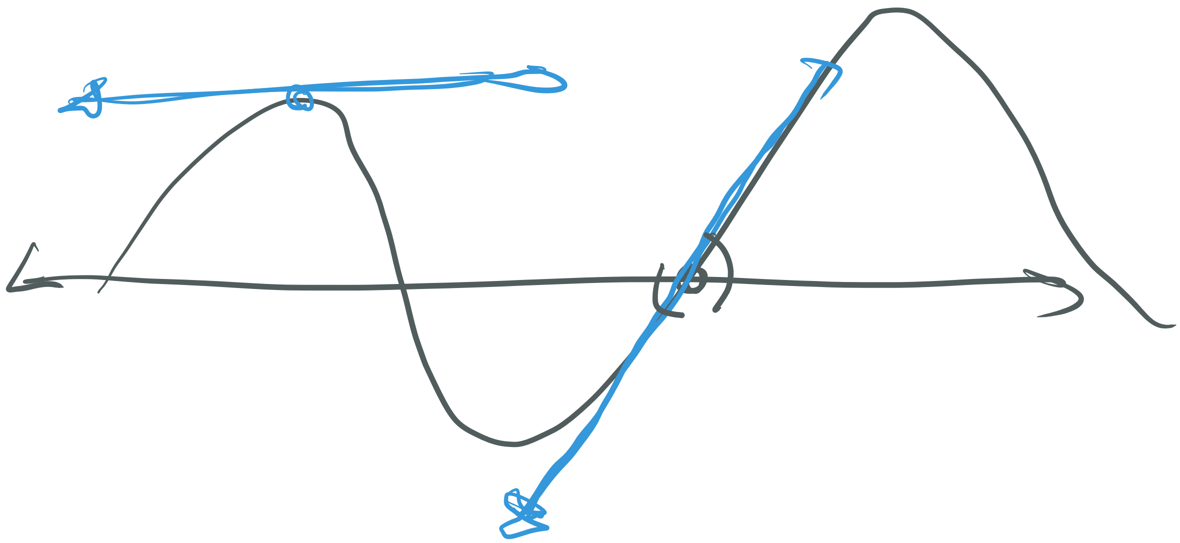
$$= \cos(0) = 1.$$

So by the limit comparison test

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so we get

that $\sum_{n=1}^{\infty} \sin(1/n)$ diverges

as well.



$\frac{1}{n} \longrightarrow 0.$

$\sin(x)$ is approximut by

$y = \underline{x}$ $\frac{1}{n}$ near $x = 0$

because $f'(0) = 1$ $f(0) = 0$

Example and Exercise

Determine if the sums converge:

$$\sum_{n=1}^{\infty} \frac{1}{(n^3+1)^{1/3}}$$

diverges

$$\sum_{n=1}^{\infty} n3^{-n}$$

converges

$$\sum_{n=1}^{\infty} \sin(1/n)$$

diverges