Absolute convergence, ratio and root tests

Alternating Series Test

The alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$

converges provided

1) the terms of the series are eventually decreasing $a_k \geq 0$

and

2) $\lim_{k \to \infty} a_k = 0$.

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$$\sum_{n=1}^{\infty}(-1)^n b_n = \sum_{n=1}^{5}(-1)^n b_n + \sum_{n=5}^{\infty}(-1)^n b_n$$

converges.

Why does the theorem work?

Assume a_k is decreasing from $k=0$

and the series is $\sum_{k=0}^{\infty} (-1)^k a_k$

$$S_0 = a_0$$

$$S_1 = a_0 - a_1$$

$$S_2 = a_0 + (\overbrace{-a_1 + a_2}^{<0})$$

$$S_3 = a_0 - a_1 + (\overbrace{a_2 - a_3}^{>0})$$

$$S_{2N+2} = S_{2N} + (\overbrace{-a_{2N+1} + a_{2N}}^{<0})$$

$$S_{2N+3} = S_{2N+1} + (\overbrace{a_{2N+2} - a_{2N+3}}^{>0}).$$

We see that S_{2N} is decreasing

and S_{2N+2} is increasing and both are bounded. Additionally

$$S_{2N+1} = S_{2N} + a_{2N+1} \quad \text{so}$$

$$\lim_{N\to\infty} S_{2N+1} = \lim_{N\to\infty} S_{2N} + \cancelto{0}{\lim_{N\to\infty} a_{2N+1}}$$

Hence,

$$\lim_{N\to\infty} S_{2N+1} = \lim_{N\to\infty} S_{2N}$$

so

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} S_{2N+1} = \lim_{N\to\infty} S_{2N}.$$

<u>Remark</u>: Let the limit be L. Then given $\varepsilon > 0$ there exist $N_1, N_2 > 0$ s.t. $\forall n_1 > N_1$, $|S_{2n_1} - L| < \varepsilon$ and $\forall n_2 \geq N_2$ $|S_{2n_2+1} - L| < \varepsilon$.

So let $N_3 = \max\{2N_1, 2N_2+1\}$. Then $\forall n_3 > N_3$ we have $|s_{n_3} - L| < \varepsilon$.

Since n_3 is either even and $n_3 > 2N_1$ or n_3 is odd and $n_3 > 2N_2+1$.

Example

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{(1+n^5)^{1/2}}$$

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If $x \geq 2$, $f'(x) \leq 0$, so f monotonic for $x \geq 2$. $\rightsquigarrow \{\frac{n^2}{(1+n^5)^{1/2}}\}$ eventually decreasing.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^{5/2}(\frac{1}{n^5}+1)^{1/2}} = \lim_{n\to\infty} \frac{1}{n^{1/2}(\frac{1}{n^5}+1)^{1/2}} = 0.$

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converges. What about $\sum(-1)^n \frac{1}{\sqrt{n}+1}$?

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Theorem: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n + |a_n| - |a_n|$$

$$= \sum_{n=0}^{\infty} a_n + |a_n| - \sum_{n=0}^{\infty} |a_n|$$

converges to some value c.

$0 \le a_n + |a_n| \le 2|a_n|$ so by the comparison tes

$\sum_{n=0}^{\infty}$ and $|a_n|$ converges as well.

Absolute convergence cont'd

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Theorem: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Converse false: $\sum (-1)^n \frac{1}{n}$ converges but not absolutely.

Examples

$$\sum \frac{\sin(n^2)}{n^3}$$

$$0 \le \left| \frac{\sin(n^2)}{n^3} \right| \le \frac{1}{n^3}$$

So

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n^2)}{n^3} \right| \le \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty.$$

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$$\sum (-1)^n \frac{1}{\sqrt{n}+1}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges

but $\frac{1}{\sqrt{n}+1}$ is decreasing and

$\lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = 0$ so converges conditionally.

Examples

$$\sum \frac{\sin(n^2)}{n^3}$$

converges absolutely.

$$\sum (-1)^n \frac{1}{\sqrt{n}+1}$$

converges conditionally, but not absolutely

Exercises

Which, if any, of the following series converge absolutely?

$$\sum \frac{(-1)^n}{16n-9}$$

$|a_n| = \frac{1}{16n-9} \sim \frac{1}{16n}$ so

Conditionally

$$\sum \frac{\cos(n\pi)}{n^3 - 1/4}$$

$|a_n| = \frac{1}{n^3 - 1/4}$

Converges absolutely

$$\sum \frac{\sin(18n)}{9^n}$$

$|a_n| = \frac{1}{9^n}$

converges absolutely

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$$\sum \frac{\cos(n\pi)}{n^3 - 1/4}$$

$$\sum \frac{\sin(18n)}{9^n}$$

1) converges conditionally, but not absolutely
2) converges absolutely
3) converges absolutely

Ratio test

Theorem: Let $\sum a_n$ be a series.

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3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, no information.

Informally this says if N is large the ratio between term a_{n+1} and a_n for $n \geq N$ is L.

So
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$$

$$= C + a_N + a_{N+1} + \cdots$$

$$= C + a_N(1 + L + L^2 + L^3 + \cdots)$$

$$= C + a_N \underbrace{\sum_{k=0}^{\infty} L^k}$$

converges if $L < 1$

diverges if $L > 1$.

<u>Remark</u>: The test for $L = 1$ is inconclusive because we really get

$$a_N \sum_{k=0}^{\infty} (L-\varepsilon)^k \leq \sum_{n=N}^{\infty} a_n \leq a_N \sum_{k=0}^{\infty} (L+\varepsilon)^k$$

for ε small. If $L<1$ we can find ε small enough so that $L+\varepsilon<1$. If $L>1$ we can find ε small enough so $L-\varepsilon>1$. If $L=1$ the $L+\varepsilon>1$ and $L-\varepsilon<1$ so the comparisons don't give a result.

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$a_n = n3^{-n}$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)3^{-n-1}}{n3^{-n}}\right| = \frac{n+1}{3n} \to \frac{1}{3} < 1$$

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$$\sum \frac{(-1)^n}{n^p}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}/(n+1)^p}{(-1)^n/n^p}\right| = \frac{n^p}{(n+1)^p} \to 1$$

no matter what p is.

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no matter what p is. $\rightsquigarrow$ no information, even though we know the answer from the p test (abs. conv. if and only if $p > 1$, conditionally for $0 < p \leq 1$).

Exercises

Determine if the following series converge absolutely, conditionally, or diverge. Use the ratio test.

$$\sum (-1)^n 9^{-n}$$

$$\sum (-1)^n \frac{9^n}{n^9}$$

$$\sum \frac{(-2)^n}{(2+n)^n}$$

1) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,9^{-(n+1)}}{(-1)^{n}\,9^{-(n)}}\right| = 9^{-1} = \frac{1}{9} < 1$$

2) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,9^{n+1}/(n+1)^9}{(-1)^n 9^n/(n)^9}\right|$$

$$= \lim_{n\to\infty}\left|9\left(\frac{(n)^9}{(n+1)^9}\right)\right| = 9 > 1$$

3) $$\lim_{n\to\infty}\left|\frac{(-2)^{n+1}/(2+(n+1))^{n+1}}{(-2)^n/(2+n)^n}\right|$$

$$= \lim_{n\to\infty} 2\left|\frac{(2+n)^n}{(2+(n+1))^{n+1}}\right|$$

$$\le \lim_{n\to\infty} 2\left|\frac{1}{2+(n+1)}\right| = 0.$$

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$$\sum (-1)^n 9^{-n}$$

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$$\sum \frac{(-2)^n}{(2+n)^n}$$

1) Converges absolutely
2) Diverges
3) Converges absolutely

Exercises

Show that $\sum \frac{r^n}{n!}$ converges for every real number r.

$$\lim_{n\to\infty} \left| \frac{r^{n+1}/(n+1)!}{r^n/n!} \right| = \lim_{n\to\infty} r \left| \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} r \left| \frac{1}{n} \right| = 0.$$

Root test

Note:

$$\sum \frac{\cos(7n)n^p}{10^n}$$

does not work for ratio test, since we have to divide by $\cos(7n)$, which might be small.

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3) If $\lim_{n\to\infty} |a_n|^{1/n} = 1$, no information.

This implies $a_n \approx L^n$. so

$$\sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{\infty} L^n$$ which converges for $L<1$

and diverges for $L>1$.

<u>Remark</u>: A similar issue to the ratio test arises with the $L=1$ case and explains why this case is inconclusive.

Exercises

Use the root test to determine if the following series converge absolutely, converge conditionally, or diverge:

$$\sum \frac{\cos(7n)n^p}{10^n}$$

$$\sum \frac{(-2)^n}{(2+n)^n}$$

we did this with the last one

but root test is easier

$$\sum (-1)^n \frac{1}{(n^3+1)^{1/2}}$$

$$\sum \left(\frac{n}{\ln(n)}\right)^n$$

① $\lim_{n\to\infty} \sqrt[n]{\left|\frac{\cos(\pi n)\, n^p}{10^n}\right|} = \lim_{n\to\infty} \frac{1}{10}\sqrt[n]{n^p}$

$= \frac{1}{10}$ Converges absolutely

② $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-2)^n}{(2+n)^n}\right|} = \lim_{n\to\infty} \frac{2}{2+n} = 0$

Converges absolutely.

③ $\lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{1}{(n^3+1)^{1/2}}\right|} = \lim_{n\to\infty} \sqrt[n]{1/(n^3+1)^{1/2}}$

$= \lim_{n\to\infty} 1/(n^3+1)^{1/2n} = 1.$

$\lim_{n\to\infty} (n^3+1)^{1/2n} = \lim_{n\to\infty} n^{\frac{3}{2n}}\left(1+\frac{1}{n^3}\right)^{1/2n}$

$= 1$

Note $\lim_{n\to\infty} \log n^{3/2n} = \lim_{n\to\infty} \frac{3}{2n}\log(n) = 0$ so $\lim_{n\to\infty} n^{3/2n} = e^0 = 1.$

Inconclusive

4 $\lim_{n\to\infty} \sqrt[n]{\left(n/\ln(n)\right)^n}$

$= \lim_{n\to\infty} \frac{n}{\ln(n)} = \infty.$ diverges

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Use the root test to determine if the following series converge absolutely, converge conditionally, or diverge:

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$$\sum \frac{(-2)^n}{(2+n)^n}$$

$$\sum (-1)^n \frac{1}{(n^3+1)^{1/2}}$$

$$\sum \left(\frac{n}{\ln(n)}\right)^n$$

1) Converges absolutely
2) Converges absolutely
3) No information, but comparison test implies it converges absolutely
4) Diverges