Table 10.4 **Special Series and Convergence Tests**

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k$, $a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

Strategy

Consider $\sum a_n$. Main idea: what does it look like?

1. Does $a_n \to 0$? If not, it diverges. If so, move on to next test.
2. Is it a specific form we know: geometric series ($a_n = r^n$), p-series ($a_n = 1/n^p$), telescoping series ($a_n = \frac{1}{n} - \frac{1}{n+1}$)?
3. Is it an alternating series? Alt series test.
4. Does it look like a p-series or a geometric series? Comparison test.
5. Does $a_n = f(n)$, with f a function you know how to integrate? Integral test.
6. Does it have factorials, or look like a geometric series but comparison didn't work? Ratio test.
7. Is $a_n = b_n^n$ for some b_n's, or does it look like that? Root test.
8. Does it ask for a remainder? Alt series test of integral test.

Getting Started

1–10. Choosing convergence tests *Identify a convergence test for each of the following series. If necessary, explain how to simplify or rewrite the series before applying the convergence test. You do not need to carry out the convergence test.*

1. $\sum_{k=1}^{\infty} (-1)^k \left(2 + \frac{1}{k^2}\right)^k$

2. $\sum_{k=3}^{\infty} \frac{2}{k^2 - k - 2}$

3. $\sum_{k=3}^{\infty} \frac{2k^2}{k^2 - k - 2}$

4. $\sum_{k=3}^{\infty} \frac{1}{k \ln^7 k}$

5. $\sum_{k=10}^{\infty} \frac{1}{(k-9)^5}$

6. $\sum_{k=10}^{\infty} \frac{100^k}{k!\, k^2}$

7. $\sum_{k=1}^{\infty} \frac{k^2}{k^4 + k^3 + 1}$

8. $\sum_{k=1}^{\infty} \frac{(-3)^k}{4^{k+1}}$

9. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2^k + \ln k}}$

10. $\sum_{k=1}^{\infty} (\tan^{-1} 2k - \tan^{-1}(2k - 2))$

Identify a convergence test for the following series. Note: There's multiple ways to solve a problem.

2) $\sum_{k=5}^{\infty} (-1)^k \left(2+\frac{1}{k^2}\right)^k$

$a_k \not\to 0$ divergence test works

Also, $a_k = b_k^k$ and

$\lim_{k\to\infty} \sqrt[k]{|-1^k \ (2+\frac{1}{k^2})^k|} = \lim_{k\to\infty} 2+\frac{1}{k^2} = 2 > 1$

root test works.

2) $\sum_{k=3}^{\infty} \frac{2}{k^2-k-2} = \sum_{k=3}^{\infty} \frac{2}{(k+1)(k-2)}$

When the denominator factors we might get a telescoping series

Telescoping $\frac{2}{k^2-k-2} = \frac{2/3}{k-2} - \frac{2/3}{k+1}$

This gives us an exact sum but not the easiest method for convergence

Comparison with $\frac{4}{k^2}$ would work

Note: $\frac{2}{k^2} \le \frac{2}{k^2-k-2}$ so a simple comparison doesn't work

Limit comparison with $\frac{1}{k^2}$ is probably the easiest

$$\lim_{k\to\infty} \frac{2/k^2-k-2}{1/k^2} = \lim_{k\to\infty} \frac{2k^2}{k^2-k-2} = 2.$$

since $1/k^2$ converges by the p-series test $2/k^2-k-2$ converges as well.

3) $\displaystyle\sum_{k=3}^{\infty} \frac{2k^3}{k^2-k-2}$

Divergence Test

$$\lim_{k\to\infty} \frac{2k^3}{k^2-k-2} = 2 \geq 0.$$

4) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k\ln^2 k}$

$\frac{1}{k} > \frac{1}{k\ln^2 k} > \frac{1}{k^2}$ so not set up for comparisons however

we can integrate

$$\int_3^{\infty} \frac{1}{x\ln^2(x)}\,dx = \int_{x=3}^{x=\infty} \frac{1}{u^2}\,du \qquad \begin{array}{l} u = \ln(x) \\ du = \frac{1}{x}dx \end{array}$$

$$= \lim_{n\to\infty} \left. \frac{-1}{6n^6} \right]_{x=3}^{x=n}$$

$$= \lim_{n\to\infty} \left[-1/6\ln^6(n) \right] - \left[-1/6\ln^6(3) \right]$$

$$= 1/6\ln^6(3) < \infty.$$

Integral test

5) $\displaystyle\sum_{k=10}^{\infty} \frac{1}{(k-9)^5}$

Integral Test works or

limit comparison with $\frac{1}{k^5}$

$$\lim_{k\to\infty} \frac{1/(k-9)^5}{1/k^5} = \lim_{k\to\infty} \frac{k^5}{(k-9)^5}$$

$$= \lim_{k\to\infty} \frac{k^5}{k^5(1-9/k)^5} = 1.$$

6) $$\sum_{k=10}^{\infty} \frac{100^k}{k!\,k^2}$$

Usually with factorials we want to use the ratio test

$$\lim_{k\to\infty} \left| \frac{100^{k+1}/((k+1)!\,(k+1)^3)}{100^k/(k!\,(k)^2)} \right| = \lim_{k\to\infty} \left| \frac{100^{k+1}\,k!\,k^2}{100^k\,(k+1)!\,(k+1)^2} \right|$$

$$= \lim_{k\to\infty} \left| \frac{100k^2}{k\cdot k^2(1+1/k)^2} \right| = \lim_{k\to\infty} \left| \frac{100}{k} \right| = 0.$$

So the series converges.

7) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k^4+k^3+1}$

Comparison test with $\frac{1}{k^2}$ works

$$\frac{k^3}{k^4+k^3+1} = \frac{1}{k^2+k+1/k^2} \le \frac{1}{k^2}$$

so the series converges since $\frac{1}{k^2}$ converges by the p-series test.

8) $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^k}{4^{k+1}}$

powers in top and bottom usually means we should simplify and use the geometric series test

$$\sum_{k=1}^{\infty} \frac{(-3)^k}{4^{2k+1}} = \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{-3}{4}\right)^k$$

$$= \frac{-3}{16}\sum_{k=0}^{\infty}\left(\frac{-3}{4}\right)^k$$

$$= \frac{-3}{16}\left(\frac{1}{1-{-3}/4}\right)$$

$$= \frac{-3}{16}\left(\frac{1}{7/4}\right) = \boxed{\frac{-3}{28}}$$

9) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2^k + \ln(k)}}$

Alternating series test

Note: If $\displaystyle\lim_{k\to\infty} \frac{1}{2^k + \ln(k)} = 0$ then

$$\lim_{k\to\infty} \frac{1}{\sqrt{2^k+\ln(k)}} = 0.$$

$$\lim_{k\to\infty} \frac{1}{2^k+\ln(k)} \le \lim_{k\to\infty} \frac{1}{2^k} = 0.$$

Therefore $\lim_{k\to\infty} \frac{1}{\sqrt{2^k+\ln(k)}} = 0.$

Note: This converges absolutely so there are other options like comparison with $\left(1/\sqrt{2}\right)^k$ to show absolute convergence.

10. $\sum_{k=1}^{\infty} \left[\tan^{-1}(2k) - \tan^{-1}(2k-2)\right]$

If we have a positive and

negative term with different indeces then we will get something telescoping.

$$\sum_{k=1}^{\infty} \tan^{-1}(2k) - \tan^{-1}(2k-2)$$

$$S_N = \left(\tan^{-1}(2) - \tan^{-1}(0)\right) + \left(\tan^{-1}(3) - \tan^{-1}(1)\right)$$

$$+ \left(\tan^{-1}(4) - \tan^{-1}(\;)\right)2 + \cdots$$

$$+ \left[\tan^{-1}(2N-2) - \tan^{-1}(2N-4)\right] + \left[\tan^{-1}(2N) - \tan^{-1}(2N-2)\right]$$