Parametric curves

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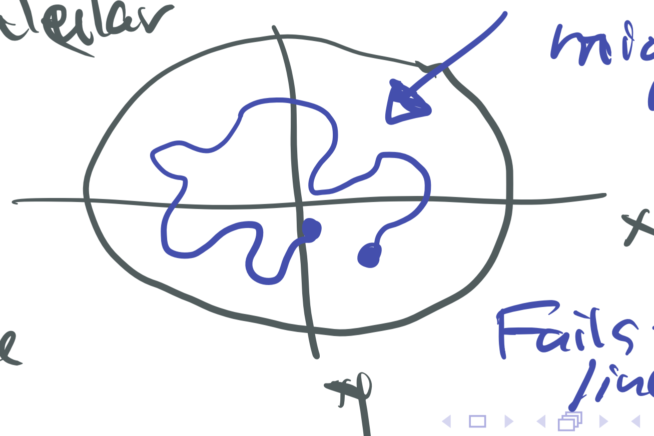
Does it move on the graph of a function?

Think of a circlular table with x,y coordinates and the origin at the center



The movement might not be a function

Fails the vertical line test.

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At time t, the x coordinate is $f(t)$ and the y coordinate is $g(t)$.

Example

Let $f(t) = 2t$, $g(t) = t^3$, where $-\infty < t < \infty$.

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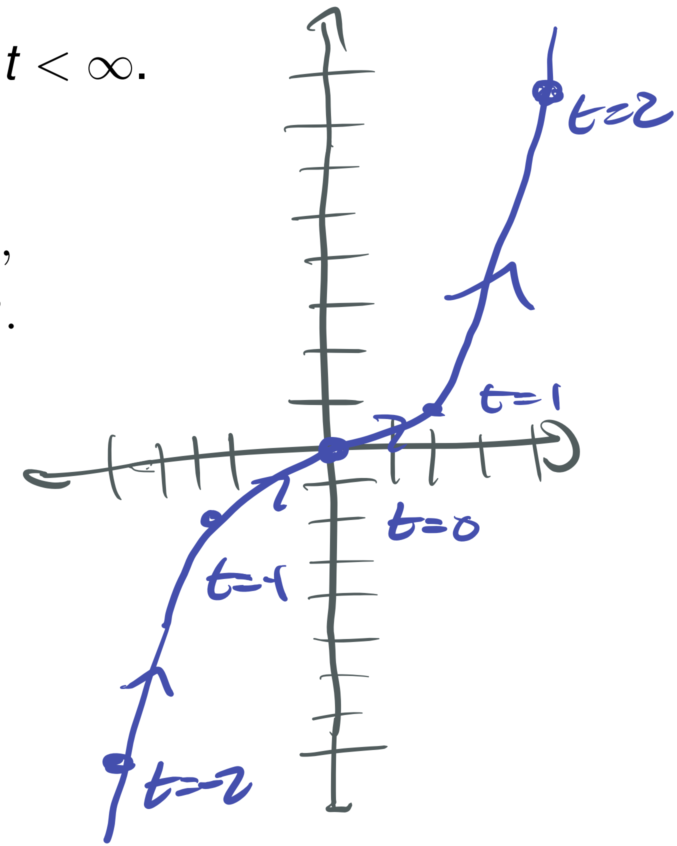
Consider parametric equations

$$\begin{cases} x = f(t) = 2t, \\ y = g(t) = t^3. \end{cases}$$

t	$f(t)$	$g(t)$
-2	-4	-8
-1	-2	-1
0	0	0
1	2	1
2	4	8



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Plot some points!
Observations: Both f and g are increasing functions. $f(t)$ is linear. That means the y coordinate is increasing as t^3 while x is growing linearly.

Continued

Special example: can solve for t: $x = 2t$ means $t = x/2$.

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Subtle: the ball rolling on this path rolls along the curve $y = x^3/8$, but it not *equal to the graph*. Need direction!

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Subtle: What do the parametric equations

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$t = \frac{x}{8}$

$y = -64\left(\frac{x}{8}\right)^3 = \frac{x^3}{8}$

Live on same *curve*, but go in opposite directions (and have different "speeds").

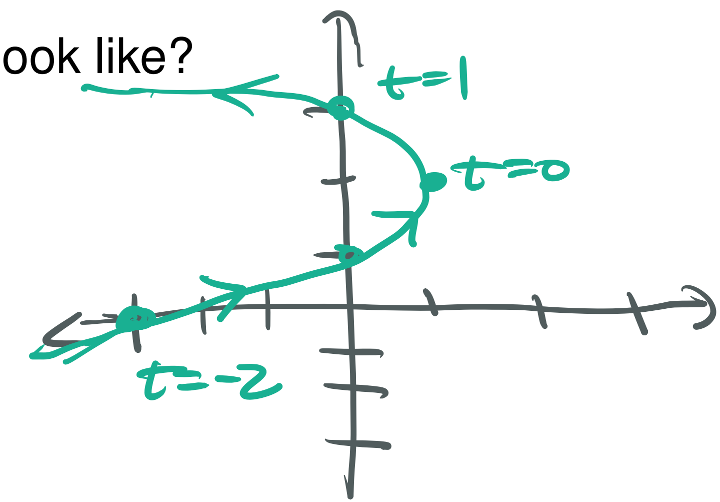
Exercise

Consider

$$\begin{cases} x = 1 - t^2, \\ y = t + 2 \end{cases}$$

for $-\infty < t < \infty$. What does it look like?

t	x	y
-2	-3	0
-1	0	1
0	1	2
1	0	3



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Lives on graph: $t = y - 2$, so $x = 1 - (y - 2)^2$.

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for $-\infty < t < \infty$. What does it look like?
Lives on graph: $t = y - 2$, so $x = 1 - (y-2)^2$. Parabola pointing to the left, with bottom at $(1, 2)$.

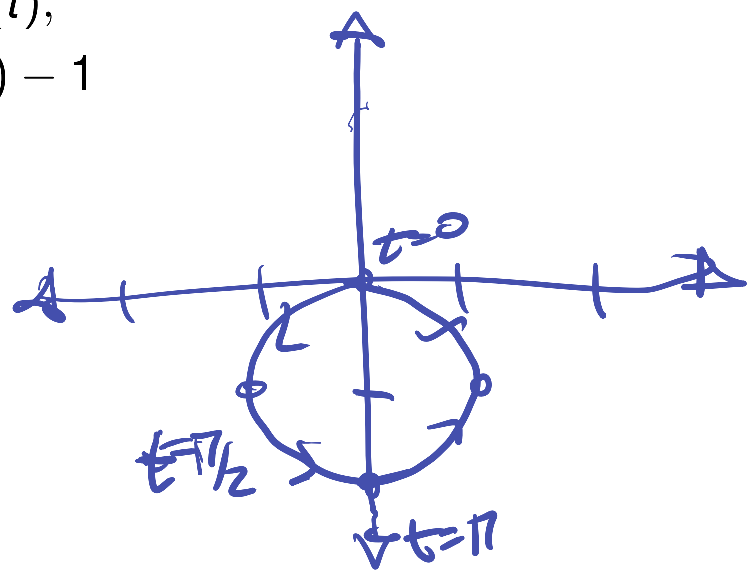
Another example

Describe the parametric motion for

$$\begin{cases} x = -\sin(t), \\ y = \cos(t) - 1 \end{cases}$$

where $0 \le t \le 4\pi$.

t	x	y
$\pi/4$	$-\sqrt{2}/2$	$\frac{\sqrt{2}}{2} - 1$
$\pi/2$	-1	-1
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2 - 1$



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$\sin(t) = -x$, $\cos(t) = y + 1$.
So

$$1 = \sin^2(t) + \cos^2(t) = x^2 + (y+1)^2,$$

which is a circle. Direction? Where does it start and where does it stop?

Exercise

Describe the curve:

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for $\pi/2 \leq t \leq 3\pi$.

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t	x	y
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$3\pi/2$	-1	-1
2π	$-3/4$	0

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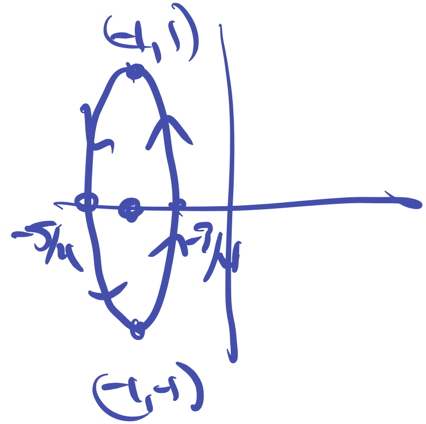

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Plot some points
$\cos(t) = 4x + 4$, $\sin(t) = y$, so lives on

$$1 = \cos^2(t) + \sin^2(t) = (4x+4)^2 + y^2 = 16(x+1)^2 + y^2$$

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Ellipse! Direction? Start/stop?

Other ways to visualize

Plot $x = f(t)$ and $y = g(t)$ on separate graphs. Try to put together.

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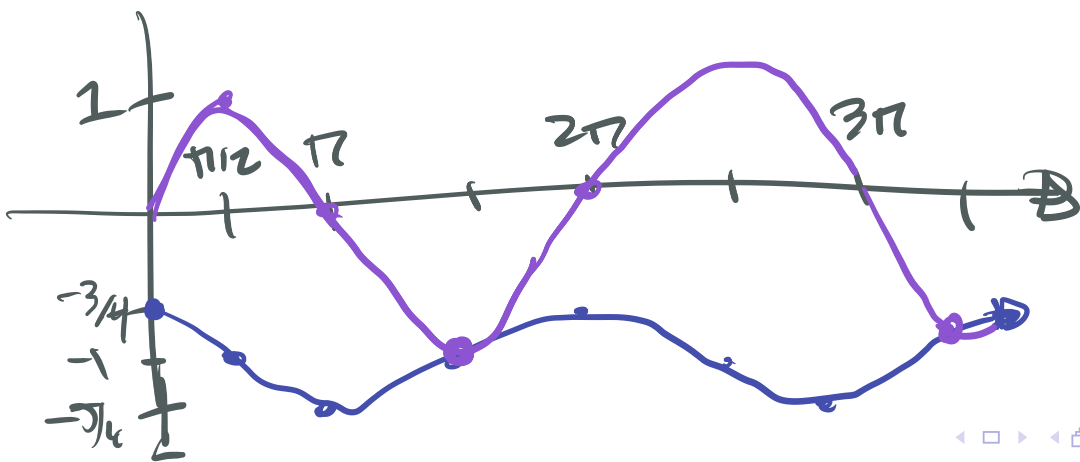
Plot $x = f(t)$ and $y = g(t)$ on separate graphs. Try to put together.
Example: $x = -\sin(t)$, $y = \cos(t) - 1$.

Exercise

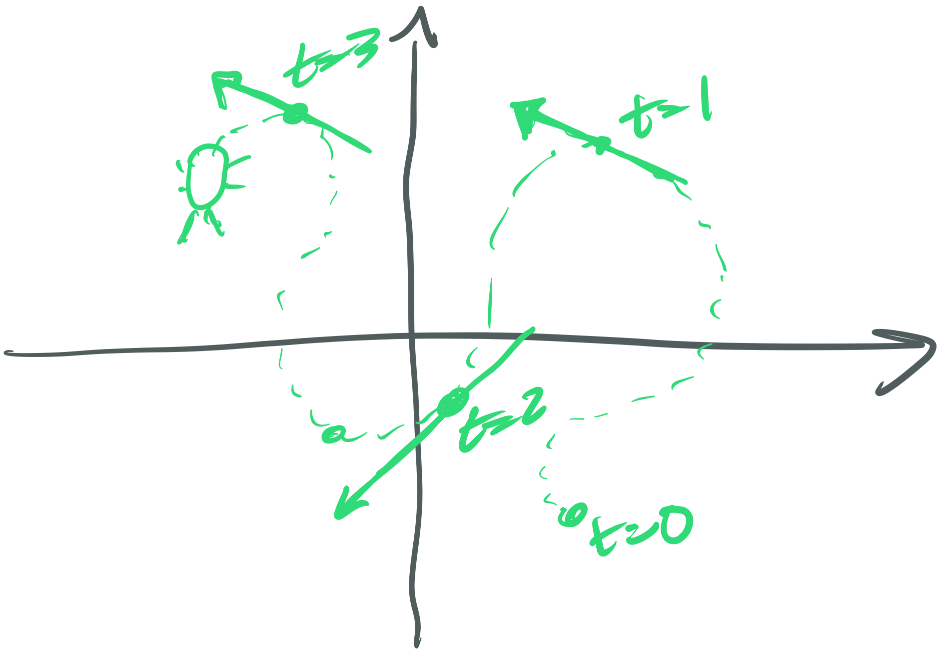
Graph x and y as functions of t to sketch the curve.

$$\begin{cases} x = \frac{1}{4}\cos(t) - 1, \\ y = \sin(t) \end{cases}$$



Calculus for parametric equations

If a cockroach is following a parametric curve, it would be nice to know the direction, or *tangent*.



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$$g(t) = F(f(t))$$

$$\frac{d}{dt}g(t) = \frac{d}{dt}F(f(t)) = F'(f(t))f'(t),$$

so *if* $f'(t) \neq 0$,

$$F'(f(t)) = \frac{g'(t)}{f'(t)}$$

Notation

This says that the *slope* of the graph where the cockroach moves is given by the rate of change in the y direction divided by the rate of change in the x direction. **Note:** Finding the slope does not require eliminating t!

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Sketch

Example: $x = 1 - t^2$, $y = t + 2$. What happens at $t = 0$?

vertical line.

$$\frac{dx}{dt} = -2t$$

$$\frac{dy}{dt} = 1$$

$$\frac{dy}{dx} = \frac{1}{-2t}$$

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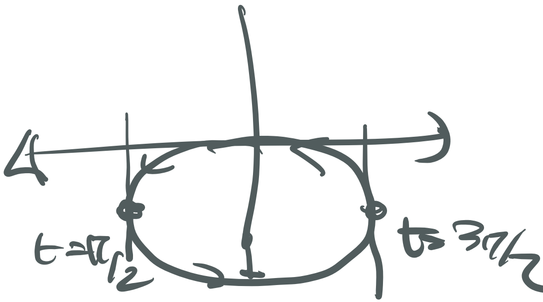

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$$\frac{dx}{dt} = -\cos(t) = 0 \text{ at } \pi/2, 3\pi/2$$

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Recall: A line is $y = mx + b$, m is slope, b is "y intercept". More useful: $(y - y_0) = m(x - x_0)$, where m is the slope at (x_0, y_0).

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$t = -1$: $m = \frac{dy}{dx} = -\frac{1}{2t}|_{t=-1} = \frac{1}{2}$. $(x_0, y_0) = (0, 1)$

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$t = 2$, $m = -\frac{1}{4}$, $(x_0, y_0) = (-3, 4)$, $\rightsquigarrow (y - 4) = -\frac{1}{4}(x + 4)$

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$t = \pi/2$, slope not defined ($x = -1$).

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$dy/dx = \tan(t)$. $t = 0$: $m = 0$, $(x_0, y_0) = (0, 0)$, $\rightsquigarrow y = 0$.
$t = \pi/2$, slope not defined ($x = -1$). $t = 3\pi/4$,
$m = \tan(3\pi/4) = -1$. $(x_0, y_0) = (-1/\sqrt{2}, -1/\sqrt{2} - 1)$
$\rightsquigarrow (y + 1 + \sqrt{2}) = -(x + \sqrt{2})$.

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$$\int_{f(a)}^{f(b)} y dx = \int_{a}^{b} g(t) f'(t) dt.$$

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Compute area enclosed by curve and $x = 0$.
$x = 0$ when $t = \pm 1$. $f'(t) = -2t$. Area = area under top curve - area under bottom curve. $\rightsquigarrow [0, 1], [-1, 0]$
Bottom curve is easier:

$$\text{Area under bottom} = \int_{x=f(-1)}^{x=f(0)} F(x)dx = \int_{t=-1}^{t=0} g(t)f'(t)dt$$
$$= \int_{-1}^{0} (t+2)(-2t)dt = 4/3$$

Continued

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Want $0 \leq x \leq 1$, which corresponds to $x = f(1) = 0$ and $x = f(0) = 1$.

$$\text{Area under top} = \int_{f(1)}^{f(0)} F(x)dx = \int_{1}^{0} g(t)f'(t)dt$$
$$= \int_{1}^{0} (t+2)(-2t)dt = \int_{0}^{1} (t+2)(2t)dt = 8/3$$

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$$= \int_{1}^{0} (t+2)(-2t)dt = \int_{0}^{1} (t+2)(2t)dt = 8/3$$

Area = $8/3 - 4/3 = 4/3$.

Exercise

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Top curve is $0 \leq t \leq \pi$. Bottom is $\pi \leq t \leq 2\pi$.

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$0 \leq t \leq 2\pi$.
Top curve is $0 \leq t \leq \pi$. Bottom is $\pi \leq t \leq 2\pi$.
Top curve goes from $x = -1/4 = f(\pi)$ to $1/4 = f(0)$, so area is

$$\int_{t=\pi}^{t=0} \sin(t)\frac{1}{4}(-\sin(t))dt = \pi/8$$

Bottom is $-\pi/8$, so sum area is $\pi/4$. This is an ellipse, so correct!

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$$\int_{x=\alpha}^{x=\beta} (1+(dy/dx)^2)^{1/2} dx = \int_{t=a}^{t=b} \left(1 + \left(\frac{(dy/dt)}{(dx/dt)} \right)^2 \right)^{1/2} (dx/dt)dt$$

Arclength

Recall *Arclength*: If $y = F(x)$ over an interval $[\alpha, \beta]$, the arclength is

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$$\rightsquigarrow L = \int_a^b ((dx/dt)^2 + (dy/dt)^2)^{1/2} dt = \int_a^b ((f')^2 + (g')^2)^{1/2} dt.$$

Example

Circle:

$$\begin{cases} x = R\cos(t), \\ y = R\sin(t) \end{cases}$$

$0 \leq t \leq 2\pi$. Circumference is $2\pi R$.

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$$\int_0^{2\pi} ((dx/dt)^2 + (dy/dt)^2)^{1/2} dt = \int_0^{2\pi} (R^2 \sin^2 t + R^2 \cos^2 t)^{1/2} dt$$

$$= \int_0^{2\pi} R dt = 2\pi R.$$

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$$\begin{cases} x = t^3, \\ y = \frac{t^2}{2}, \end{cases}$$

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$$= \int_1^3 (9t^4 + t^2)^{1/2} dt = \int_1^3 t(9t^2 + 1)^{1/2} dt$$

u sub $u = 9t^2 + 1$, $du = 18tdt$, so

$$\int t(9t^2 + 1)^{1/2} dt = \frac{1}{18} \int u^{1/2} du = \frac{1}{27} u^{3/2} = \frac{1}{27} (9t^2 + 1)^{3/2}$$

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Plugging in, $L = \frac{1}{27}((9^3 + 1)^{3/2} - (9 + 1)^{3/2})$

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$$L = \int_1^2 ((2t - 4t^3)^2 + (4\sqrt{2}t^2))^{1/2} dt$$

$$= \int_1^2 (4t^2 - 16t^4 + 16t^6 + 32t^4)^{1/2} dt$$

$$= \int_1^2 2t(1 + 4t^2 + 4t^4)^{1/2} dt = \int_1^2 2t(1 + 2t^2) dt$$

$$= (t^2 + t^4)|_1^2 = (4 + 16) - 2 = 18$$

Surface area

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Circumference arclength

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$$S = 2\pi \int_0^\pi (R\sin(t))((-R\sin(t))^2 + (R\cos(t))^2)^{1/2} dt$$
$$= 2\pi \int_0^\pi R^2 \sin(t) dt = 4\pi R^2$$

Exercise: torus

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$$\begin{aligned} S &= 2\pi \int_0^{2\pi} (r\sin t + R)((-r\sin t)^2 + (r\cos t)^2)^{1/2} dt \\ &= 2\pi \int_0^{2\pi} (r\sin(t) + R) r dt \\ &= 2\pi r(-r\cos t + Rt)|_0^{2\pi} \\ &= 4\pi^2 rR. \end{aligned}$$