Slope of tangent lines

Recall a polar equation is of the form $r = f(\theta)$. Now, $x = r\cos\theta = f(\theta)\cos(\theta)$

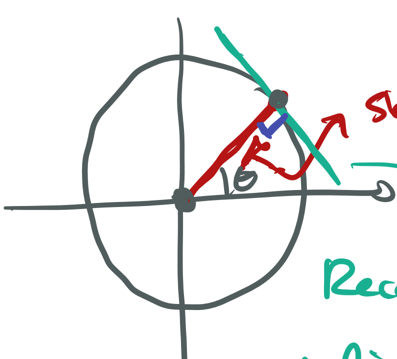
$y = r\sin\theta = f(\theta)\sin(\theta)$

So, this is like a parametric equation but in the variable θ instead of t.

So, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Example: Find the slope of the lines tangent to the circle $r=1$.

$r = f(\theta) = 1$.



slope of this line is $\tan\theta = \frac{\sin\theta}{\cos\theta}$

This is the line tangent

Recall if a line l_1 is perpendicular to a line l with slope m then the slope of l_1 is $-1/m$. Hence

The slope of this line is $\frac{-\cos\theta}{\sin\theta} = -\cot\theta$.

Using the formula let's check this works

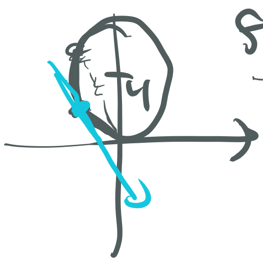
$$\text{slope} = \frac{\overset{=0}{f'(\theta)}\sin\theta + \overset{=1}{f(\theta)}\cos\theta}{\underset{=0}{f'(\theta)}\cos\theta - \underset{=1}{f(\theta)}\sin\theta} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

$r = 1$

Example:

Compute the slope of the tangent line to $r = 8\sin\theta$ at $\left(4, \frac{5\pi}{6}\right)$

$f'(\theta) = 8\cos\theta$



$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

$$= \frac{8\cos\theta\sin\theta + 8\cos\theta\sin\theta}{8\cos^2\theta - 8\sin^2\theta}$$

$$= \frac{\overset{2}{\cancel{16}}\cos\theta\sin\theta}{\cancel{8}[\cos^2\theta - \sin^2\theta]}$$

$$= \frac{2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\left(-\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

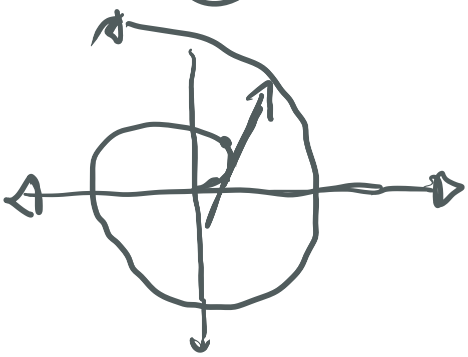
$= \boxed{-4\sqrt{3}}$

$f(\frac{\pi}{4}) = \frac{\pi}{2}$

$r = 2\theta$

__Exercise:__

Compute the slope of the curve $r = 2\theta$ at $(\pi/2, \pi/4)$



$\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \frac{f'(\frac{\pi}{4})\cancel{\cos}(\frac{\pi}{4}) + f(\frac{\pi}{4})\cancel{\cos}(\frac{\pi}{4})}{f'(\frac{\pi}{4})\cos\cancel{(\frac{\pi}{4})} - f(\frac{\pi}{4})\cancel{\cos(\frac{\pi}{4})}}$$

$f(\theta) = 2\theta$

$f'(\theta) = 2$

$$= \frac{f'(\frac{\pi}{4}) + f(\frac{\pi}{4})}{f'(\frac{\pi}{4}) - f(\frac{\pi}{4})} \neq 1.$$

$$\frac{dy}{dx} = \frac{2\frac{\sqrt{2}}{2} + \frac{\pi}{2}(\frac{\sqrt{2}}{2})}{2\frac{\sqrt{2}}{2} - \frac{\pi}{2}(\frac{\sqrt{2}}{2})}$$

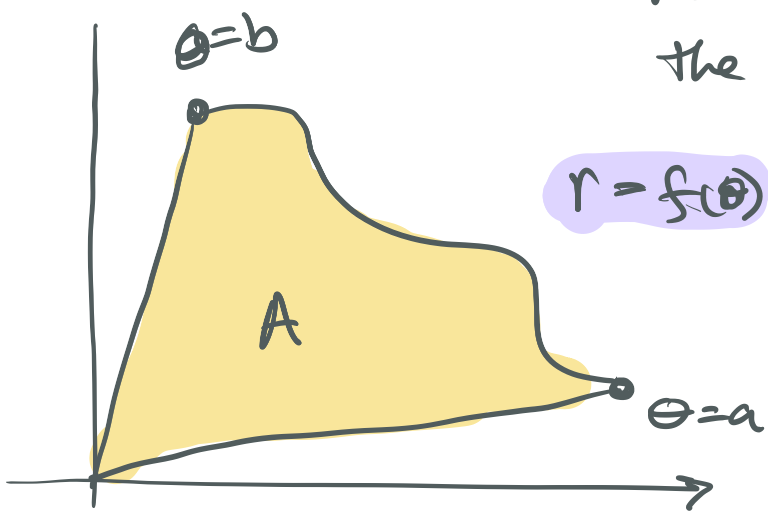
$$= \frac{\sqrt{2} + \frac{\pi\sqrt{2}}{4}}{\sqrt{2} - \frac{\pi\sqrt{2}}{4}}$$

$$= \frac{2 + \frac{\pi}{2}}{2 - \frac{\pi}{2}}$$

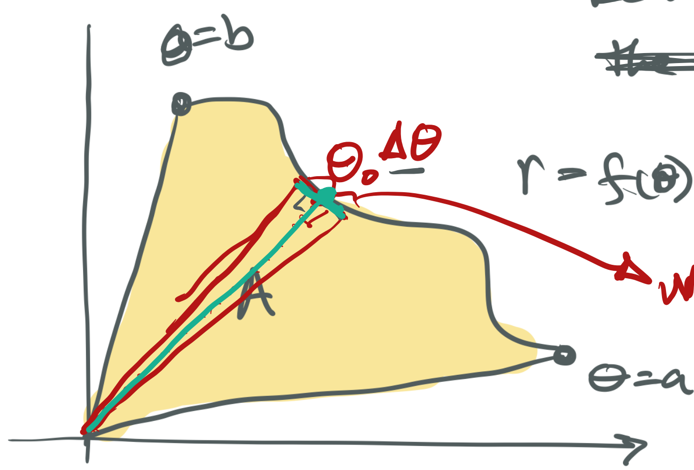
$$= \boxed{\frac{4+\pi}{4-\pi}} \approx 8$$

Areas of region bounded by Polar Curves

we want to find the area of the yellow shape



we want approximate the yellow by using ~~we want~~ - triangles ~~the area~~ ~~we~~



We draw this triangle so that the line at θ_0 is perpendicular to the radius so we know the height is $f(\theta_0)$.

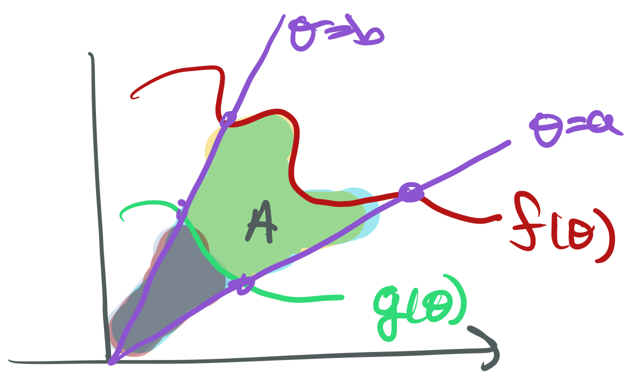
the base is then $r\Delta\theta$

So to approximate the area we split it up into a bunch of Triangles

$$\text{Area} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2} \underbrace{f(\theta_k^*)}_{\text{height}} \underbrace{f(\theta_k^*)\Delta\theta}_{\text{base}}$$

$$= \int_{\theta=a}^{\theta=b} \frac{1}{2} f(\theta)^2 \, d\theta.$$

We can extend This idea to area between curves.



$0 \le g(\theta) \le f(\theta)$

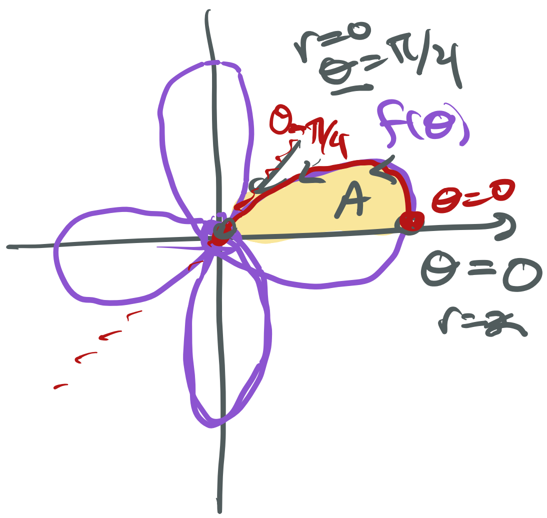
$$\text{Area} = \int_a^b \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta.$$

$$= \int_0^b \frac{1}{2} f(\theta)^2 d\theta - \int_a^b \frac{1}{2} g(\theta)^2 d\theta$$

Example: Find the area of the four-leaf rose $r = f(\theta) = 2\cos(2\theta)$

when $\cos(2\theta) = 0$, $2\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$

Even though this drawing is bad there is symmetry



$$\text{Area} = 8(\text{Area of } A)$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}}$$

$$\text{Area} = 8\int_0^{\pi/4} \frac{1}{2}\left(2\cos(2\theta)\right)^2 d\theta$$

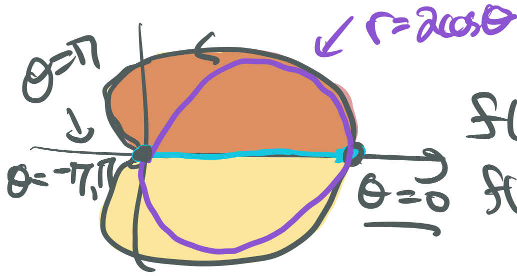
$$= 16\int_0^{\pi/4} \cos^2(2\theta)\, d\theta$$

$$= 16\int_0^{\pi/4} \frac{1}{2} + \frac{\cos(4\theta)}{2}\, d\theta$$

$$= 16\left[\frac{1}{2}\theta + \frac{\sin(4\theta)}{8}\right]_0^{\pi/4}$$

$$= 16\left[\left(\frac{\pi}{8}+0\right) - (0+0)\right] = \boxed{2\pi}$$

Exercise: Find the area of the cardioid $r = 1 + \cos\theta \to$ $r = 0$ when $\cos\theta = -1$, $\theta = \pi$



Area $= \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta$

$f(0) = 2$

$f(\pi) = 0$

$= \int_a^b f(\theta)^2 \, d\theta$

$$= \int_0^{\pi} (1 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi} \underbrace{1 + 2\cos\theta} + \underbrace{\cos^2\theta} \, d\theta$$

$$= \left[\theta + 2\sin\theta\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} 1 + \cos(2\theta) \, d\theta$$

$$\left[(\pi + 0) - (0 + 0)\right] + \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi}$$

$$= \pi + \frac{1}{2}\left[(\pi + 0) - (0 + 0)\right]$$

$$= \boxed{\frac{3\pi}{2}}$$

Arclength of Polar Curves

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

recall $x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

gives us

parametric equations so we can use that to calculate arclength

$$\text{Arclength} = \int_{\theta=a}^{\theta=b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta\right)^2 - 2f(\theta)f'(\theta)\cos\theta\sin\theta + f(\theta)^2\sin^2\theta$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta)\sin\theta\right)^2 + 2f(\theta)f'(\theta)\cos\theta\sin\theta + f(\theta)^2\cos^2\theta$$

$$f'(\theta)^2\cos^2\theta + f'(\theta)^2\sin^2\theta = f'(\theta)^2\left(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}\right)$$

So,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = f'(\theta)^2 + f(\theta)^2$$

So,

$$\text{Arc length} = \int_a^b \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

Example: Show that the circumference of the circle $r=2\cos\theta$ is 2π

The circle of radius 1 centered at $(1,0)$

$$\text{Arclength} = \int_0^{\pi} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

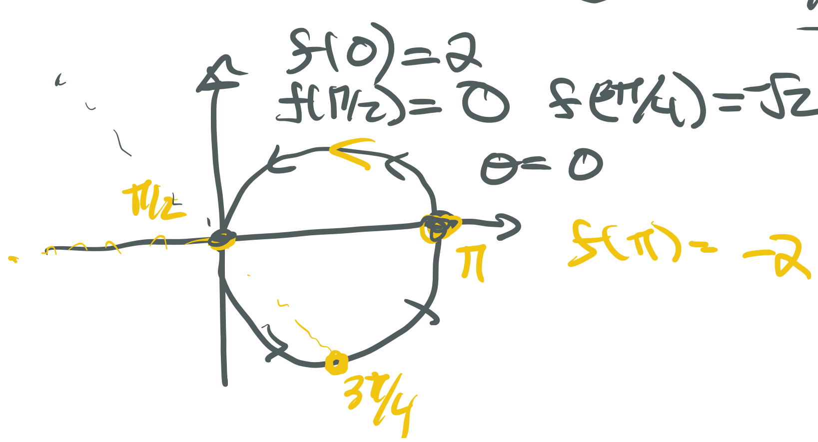
$$f'(\theta) = -2\sin\theta \qquad = \int_0^{\pi} \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{4\sin^2\theta + 4\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi} 2\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta$$

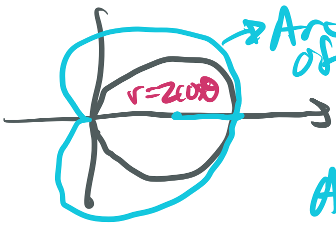
$$= \int_0^{\pi} 2\, d\theta$$

$$= \left[2\theta\right]_0^{\pi} = \boxed{2\pi}$$



Exercise: Find the arclength of the cardioid $r = 1+\cos\theta$.

Hint: Use $\cos^2(\theta/2) = \frac{1+\cos(\theta)}{2}$



Arclength of cardioid > circumference of a circle

$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}$

Arclength > 6

$$\boxed{2\int_0^{\pi} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta}$$

$$\int_0^{2\pi} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

$f'(\theta) = -\sin\theta$

$f(\theta) = 1+\cos\theta$

$$\sqrt{f'(\theta)^2 + f(\theta)^2} = \sqrt{\sin^2\theta + (1+\cos(\theta))^2}$$

$$= \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}$$

$$= \sqrt{2+2\cos\theta}$$

$$= \sqrt{4\left[\frac{1+\cos\theta}{2}\right]}$$

$$= 2\sqrt{\frac{1+\cos\theta}{2}}$$

$$= 2\cos(\theta/2)$$

$$\text{Arclength} = 2\int_0^{\pi} 2\cos(\theta/2)\,d\theta$$

$$= 4\int_0^{\pi} \cos(\theta/2)\,d\theta$$

$$= 8\left[\sin(\theta/2)\right]_0^{\pi}$$

$$= \boxed{8}$$

$f(x) = e^x$, $f^{(n)}(x) = e^x$ $f^{(n)}(0) = e^0 = 1$

Power series at $a=0$ is

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)\, x^k}{k!} \qquad f^{(10)}(0) = \frac{1}{10!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{3^n}{(n+1)!}(x-2)^n = \sum_{n=0}^{\infty} \frac{3^n}{n+1} \left(\frac{(x-2)^n}{n!}\right)$$

$$f^{(10)}(2) = \frac{3^{10}}{11!}$$

$f^{(n)}(2)$

$$f^{(10)}(2) = \boxed{\frac{3^{10}}{11}}$$

Taylor series formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$x^3\left(\frac{1}{1-x}\right) = x^3\left(\sum_{k=0}^{\infty} x^k\right) \longrightarrow \sum_{k=0}^{\infty} x^{k+3}$$

$$\frac{+1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$$

$$\ln(1-x) = \sum_{k=0}^{\infty} \int x^k \, dx$$

$$= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

e^x

$\frac{d}{dx} e^x = e^x$

$\sin(2x)$ $\quad a=0.$

$f^{(0)}(0) = \sin(0) = 0.$

$$f^{(1)}(x) = 2\cos(2x) \quad f'(0) = \underline{2} = 2^1$$

$$f^{(2)}(x) = -\underline{4}\sin(2x) \quad \underline{f''(0)} = 0$$

$$f^{(3)}(x) = -8\cos(2x) \quad f'''(0) = -8 = \underline{(-1)} \cdot 2^3$$

$$f^{(4)}(x) = +16\sin(2x) \quad f^{(4)}(0) = 0.$$

$$f^{(5)}(x) = 32\cos(2x) \quad f^{5}(0) = 32 = \underline{2^5}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

$$= \frac{2^1 x^1}{1!} + \frac{-2^3 x^3}{3!} + \frac{2^5 x^5}{5!}$$

$$= \frac{2x}{1!} - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!}$$

$k=0 \quad k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5$

odd numbers $\quad \underline{2k+1}$

$$\sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1} x^{2k+1}}{(2k+1)!} = \sin(\underline{2}x)$$

$$\overset{f(x)}{\underline{\sin}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$f(x) = 2x$

$$\sin(f(x)) = \sum_{k=0}^{\infty} \frac{(-1)^k (f(x))^{2k+1}}{(2k+1)!}$$

$$\sin(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k+1}}{(2k+1)!}$$

$$f(x) = \frac{x}{1+9x^2}$$

$$-x^5 + \frac{x^7}{\underset{2^3}{8}} - \frac{x^9}{\underset{3^3}{27}} + \frac{x^{11}}{\underset{4^3}{64}} -$$

$k=1$

powers increasing

$$= \frac{x^5}{1^3} + \frac{x^7}{2^3} - \frac{x^9}{3^3} + \frac{x^{11}}{4^3}$$

$$(-1)^k \frac{x^{2k+3}}{(k)^3}$$

but this isn't quite correct.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k} x^{2k+3}}{k^3}$$

$|x|<1$

$x=1$

At $x=1$ $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges

converges absolutely by the p-series test.

$x=(-1)$

Interval ~~Radius~~ of convergence is $[-1,1]$.

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$$

interval of convergence

$[-1,1]$