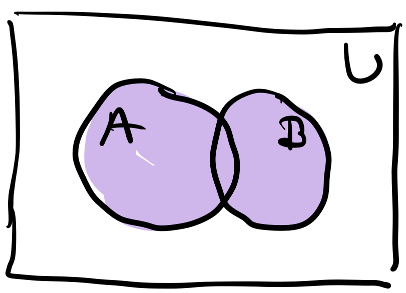
## 2.2 Set Operators

### Definition 1

Let A and B be sets. The <u>union</u> of the sets A and B, denoted by $A \cup B$, is the set that contains those elements in A or in B, or both.

$$\boxed{A \cup B := \{x \mid x \in A \vee x \in B\}}$$




$\blacksquare = A \cup B$

Example:

The union of the sets $\{1,3,5\}$ and $\{1,2,3\}$ is the set $\{1,2,3,5\}$

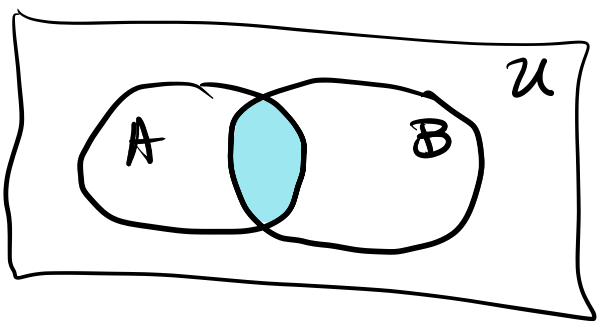
$$\{1,3,5\} \cup \{1,2,3\} = \{1,2,3,5\}$$

Remember $\{1,2,3,3,5\} = \{1,2,3,5\}$.

## Definition 2

Let $A$ and $B$ be sets. The <u>intersection</u> of the sets $A$ and $B$, denoted by $A \cap B$, is the set containing those elements in both $A$ and $B$.

$$\boxed{A \cap B = \{x \mid x \in A \wedge x \in B\}}$$




■ $= A \cap B$

**Example:**

The intersection of the sets $\{1,2,3\}$ and $\{1,3,5\}$ is the set $\{1,3\}$.

$$\{1,2,3\} \cap \{1,3,5\} = \{1,3\}.$$

## Definition 3

Two sets $A$ and $B$ are <u>disjoint</u> if the intersection of $A$ and $B$ is the empty set.

(A and B are disjoint $\iff A \cap B = \emptyset$).

Example

$A = \{1, 3, 5, 7, 9\} = \{x \in \mathbb{Z}^+ \mid x \leq 10 \wedge x \text{ is odd}\}$

$B = \{2, 4, 6, 8, 10\} = \{x \in \mathbb{Z}^+ \mid x \leq 10 \wedge x \text{ is even}\}$

$A \cap B = \emptyset$ so A and B are disjoint.

Remark: This will be proved in Chapter 6 but be aware $|A \cup B| \neq |A| + |B|$.

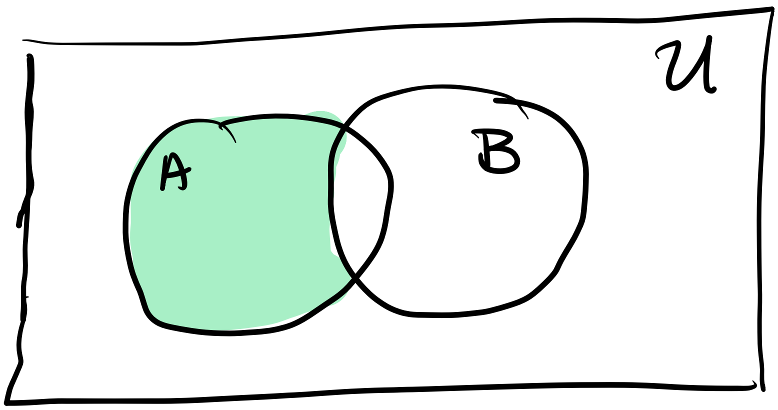
$|A \cup B| = |A| + |B| - |A \cap B|$.

Definition 4.

Let A and B be sets. The difference of A and B, denoted $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is called the complement of B with respect to A.

<u>Note:</u> I will use the notation $A \setminus B$ for difference most of the time.

$$A \setminus B = \{ x \in A \mid x \notin B \}.$$
$$= \{ x \mid x \in A \land x \notin B \}.$$



$\square = A \setminus B.$

<u>Example:</u>

$\{1, 3, 5\} \setminus \{1, 2, 3\} = \{5\}$

$\{1, 2, 3\} \setminus \{1, 3, 5\} = \{2\}$

$A \setminus B \neq B \setminus A$ in general.

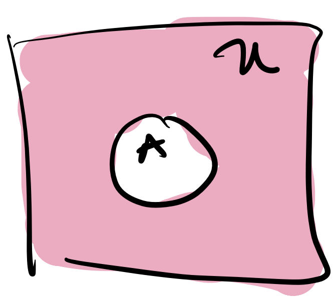
## Definition 5

Let $\mathcal{U}$ be the universal set. The complement of the set $A$, denoted by $\bar{A}$, is the complement of $A$ with respect to $\mathcal{U}$. Therefore $\bar{A}$ is $\mathcal{U} \setminus A$.

**Note:** Please use the notation $A^c$ for complement.

**Remark:** The definition of $A^c$ depends on a particular universal set. This definition makes sense for any superset $\mathcal{U}$ of $A$. Usually the set $\mathcal{U}$ is assumed for example $\mathbb{R}$ or $\mathbb{R}^2$ in analysis or $\mathbb{C}$ in complex analysis.

$$A^c = \{x \in \mathcal{U} \mid x \notin A\}$$




$= A^c = \mathcal{U} \setminus A.$

Example:

Let $A = \{a, e, i, o, u\}$ and $\mathcal{U}$ be the set of english letters.

$$A^c = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$$

Let $A = \{x \in \mathbb{Z}^+ \mid x > 10\}$ and $\mathcal{U} = \{x \in \mathbb{Z}^+\}$

Then $A^c = \{x \in \mathbb{Z}^+ \mid x \leq 10\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Exercise 21

$$A \setminus B = A \cap \overline{B}.$$

**TABLE 1 Set Identities.**

| *Identity* | *Name* |
|---|---|
| $A \cap U = A$<br>$A \cup \varnothing = A$ | Identity laws |
| $A \cup U = U$<br>$A \cap \varnothing = \varnothing$ | Domination laws |
| $A \cup A = A$<br>$A \cap A = A$ | Idempotent laws |
| $\overline{(\overline{A})} = A$ | Complementation law |
| $A \cup B = B \cup A$<br>$A \cap B = B \cap A$ | Commutative laws |
| $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$ | Associative laws |
| $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | Distributive laws |
| $\overline{A \cap B} = \overline{A} \cup \overline{B}$<br>$\overline{A \cup B} = \overline{A} \cap \overline{B}$ | De Morgan's laws |
| $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$ | Absorption laws |
| $A \cup \overline{A} = U$<br>$A \cap \overline{A} = \varnothing$ | Complement laws |

# Proving set Equalities

**Method 1:** Two way containment

Showing $A=B$ by showing (i) $A \subseteq B$ and (ii) $B \subseteq A$

**Method 2:** Set builder proofs

Showing $A=B$ by starting with $A = \{x \mid P(x)\}$ and changing $P(x)$ through rules of logic and theorem/def. to get $Q(x)$ where $B = \{x \mid Q(x)\}$.

$$A^c = \{x \in \mathbb{Z} \mid x \text{ is odd}\}^c = \{x \in \mathbb{Z} \mid \neg(x \text{ is odd})\}$$

$$= \{x \in \mathbb{Z} \mid x \text{ is even}\} = B.$$

Method 3: Successive Substitution using Table 1.

$$\overline{A \cap B \cap C} = \overline{A \cap B} \cup \overline{C} \quad \text{DeMorgan's Law}$$

$$= (\overline{A} \cup \overline{B}) \cup \overline{C} \quad \text{DeMorgan's Law}$$

$$= \overline{A} \cup \overline{B} \cup \overline{C} \quad \text{Associative Law}$$

---

Example:

Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$ using a two way containment proof.

(i) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A \cap B}$. Then $x \notin A \cap B$. Hence $x \notin A$ or $x \notin B$. So, $x \in \overline{A}$ or $x \in \overline{B}$. Therefore $x \in \overline{A} \cup \overline{B}$

(ii) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Suppose $x \in \overline{A} \cup \overline{B}$

Then $x \in \overline{A}$ or $x \in \overline{B}$. So, $x \notin A$ or $x \notin B$. Thus, $x \notin A \cap B$. Therefore $x \in \overline{A \cap B}$

Since $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ and $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$. ■

<u>Example</u>

Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$ using set builder <u>notation</u>.

$\overline{A \cap B} = \{x \in U \mid x \notin A \cap B\}$ Definition of Complement

$= \{x \in U \mid \neg(x \in A \cap B)\}$ Def. on negation

$= \{x \in U \mid \neg(x \in A \wedge x \in B)\}$ def of intersection

$= \{x \in U \mid x \notin A \vee x \notin B\}$ DeMorgan's Law

$= \{x \in U \mid x \in \overline{A} \vee x \in \overline{B}\}$ Definition of Complement.

$= \overline{A} \cup \overline{B}$. Definition of Union

## Example:

Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$ using successive substitution

$$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)} \quad \text{Demorgan's Law}$$

$$= \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{Demorgan's law}$$

$$= (\overline{B} \cup \overline{C}) \cap \overline{A} \quad \text{Commutative Law}$$

$$= (\overline{C} \cup \overline{B}) \cap \overline{A} \quad \text{Commutative law}$$

## Definition 6

The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of $A_1, A_2, \dots, A_n$.

## Definition 7

The <u>intersection</u> of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of $A_1, A_2, \ldots, A_n$

### Example

For $i = 1, 2, \ldots$ let $A_i = \{i, i+1, i+2, \ldots\}$

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\} = \mathbb{N} \setminus \{0\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, \ldots\} = A_n.$$

---

We can extend this to countably and uncountably infinite intersections and unions.

Given sets $A_1, A_2, \dots, A_n, \dots$ Then

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \in \mathbb{Z}^+, x \in A_i\}$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \in \mathbb{Z}^+, x \in A_i\}$$

Example: Let

$A_i = [\frac{1}{i}, \infty)$. Then,

$$\bigcup_{i=1}^{\infty} A_i = (0, \infty).$$

Now this doesn't have to be countable.

In general,

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I, x \in A_i\}$$

Where $I$ is a set. For example let

$I = \mathbb{R}^+$, $A_i = [\frac{1}{i}, \infty)$

$$\bigcup_{i \in I} A_i = (0, \infty) = \bigcup_{i=1}^{\infty} A_i$$

However, these are not the unions of the same sets.