2.3 Functions

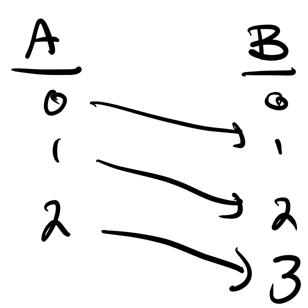
Definition 1

Let A and B be nonempty sets. A function from A to B is an assignment of exactly one element of B to each element of A. We write $f(a)=b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.

Remark: Functions are sometimes called mappings or transformations.

There are many ways to define functions

Let $A = \{0, 1, 2\}$ $B = \{0, 1, 2, 3\}$ and f is a function from A to B.



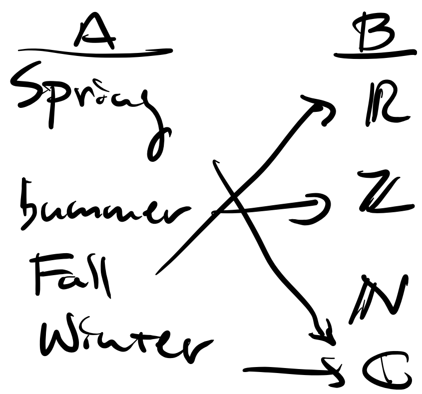
x	$f(x)$
0	1
1	2
2	3

$f(x) = x+1$

All define the same function.

Remark: Functions can be from any set A to a set B and the connection between x and $f(x)$ doesn't need to make sense.

Let A be the set of seasons and $B = \{\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{C}\}$. Let f be a function from A to B defined by



This is a well defined function.

Another way to define a function $f: A \to B$ is through a relation from A to B. A relation from A to B that contains one and only one ordered pair (a, b), $\forall a \in A$ defines a function $f: A \to B$. The set $\{(a, f(a)) : a \in A\}$ is the graph of f.

Definition 2

If f is a function from A to B, we say that A is the domain of f and B is the codomain of f. If $f(a)=b$, we say that b is the image of a and a is the preimage of b. The range or image of f is the set of all images of elements of A. We say f maps A to B.

Definition:

Functions f and g are equal if the have the same domain, codomain and map each element in the domain to the same element in the codomain.

$f: \mathbb{Z} \to \mathbb{Z}$ $f(x)=x$ and

$g: \mathbb{Z} \to \mathbb{R}$ $g(x)=x$ are not equal since the codomain is different.

Example:

Let $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$. Then Image of f is the set $\{0, 1, 4, 9, 16, \ldots\}$. which is the set of all perfect squares.

Definition 3

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then, $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined $\forall x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{and}$$

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Example

Let f_1, f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x$ and $f_2(x) = x - x^2$. Then

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x + x - x^2 = 2x - x^2$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) = x(x - x^2) = x^2 - x^3.$$

Definition 4

Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$ so that

$$f(S) = \{t \mid \exists s \in S\, (t = f(s))\}$$
$$= \{f(s) \mid s \in S\}.$$

Remark:

$f(A)$ is the image of f.

Definition 5

A function f is one-to-one or injective if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

Remark: this is equivalent to
$\forall a \forall b (f(a) = f(b) \to a = b)$ and
$\forall a \forall b (a \neq b \to f(a) \neq f(b))$.

Example

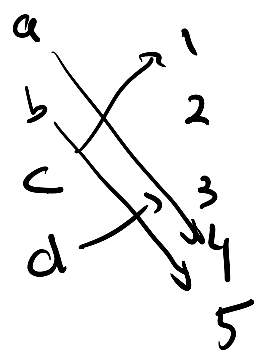
Let f be a function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with

$f(a) = 4$

$f(b) = 5$

$f(c) = 1$

$f(d) = 3$



This is injective since only one element in a is mapped to an element in b. At most one arrow points to each element in B.

Example:

Is the function that maps each person in this class to their age injective.

(Note: $A = \{\text{students in this class}\}$ $B = \mathbb{N}$) ← or could be a smaller set.

No, two people have the same age.

Is the function $f: \mathbb{Z} \to \mathbb{Z}$ s.t. $f(x) = x^2$ injective?

No, $f(-1)=(-1)^2=1^2=f(1)$ but $-1\neq 1$.

Is the function $f:\mathbb{Z}^+\to\mathbb{Z}$ s.t. $f(x)=x^2$ injective?

Yes, if $f(a)=f(b)$ then $a^2=b^2$. Since $a,b>0$ we must have $a=b$.

Is the function $f:\mathbb{R}\to\mathbb{R}$ s.t. $f(x)=x+1$ injective?

Yes, if $f(a)=f(b)$ then $a+1=b+1$. So, $a=b$.

<u>Definition 6</u>

A function f whose domain and codomain are subsets of the sets of real numbers is called <u>increasing</u> if $f(x)\leq f(y)$ and <u>strictly increasing</u> if $f(x)<f(y)$ whenever $x<y$ and x and y are in the domain of f.

A function f whose domain and codomain are subsets of the sets of real numbers is called <u>decreasing</u> if $f(x) \geq f(y)$ and <u>strictly decreasing</u> if $f(x) > f(y)$ whenever $x < y$ and x and y are in the domain of f.

<u>Remarks</u>:

A function is increasing $\Leftrightarrow \forall x^{\in A} \forall y^{\in A} (x > y \to f(x) \geq f(y))$

A function is decreasing $\Leftrightarrow \forall x^{\in A} \forall y^{\in A} (x > y \to f(x) \leq f(y))$.

<u>Example</u>

The function $f(x) = x^2$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ is strictly increasing. And $f(x) = x^2$ from $\mathbb{R}^-$ to $\mathbb{R}^-$ is strictly decreasing.

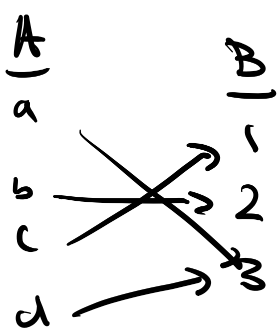
<u>Definition 7</u>

A function f from A to B is called onto or <u>surjective</u> if and only if for every element of $b \in B$ there is an $a \in A$ such that $f(a) = b$.

f is onto $\iff \forall b \in B \, \exists a \in A \, (f(a) = b)$.

Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. let f be a function from A to B defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$



All 3 elements in B have some element mapped to it. Every element in B has some arrow pointing to it.

Example

Is the function $f: \mathbb{Z} \to \mathbb{Z}$ s.t. $f(x) = x^2$ surjective?

No, $\sqrt{2} \in \mathbb{Q}$ so there is no $x \in \mathbb{Z}$ s.t. $x^2 = 2$.
Also, $x^2 \geq 0$ so there is no $x \in \mathbb{Z}$ s.t. $x^2 = -1$.

Is the function $f: \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ s.t. $f(x) = x^2$ surjective?

yes, Given $y \in \mathbb{R}^+ \cup \{0\}$. $\sqrt{y} \in \mathbb{R}$. hence $f(\sqrt{y}) = (\sqrt{y})^2 = y$.

Is the function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x^2$ surjective? No, $x^2 \geq 0$ if $x \in \mathbb{R}$ so there is no $x \in \mathbb{R}$ s.t. $x^2 = -1$.

<u>Remark</u>: $f: \mathbb{C} \to \mathbb{C}$ s.t. $f(x) = x^2$ is surjective.

The codomain is important!

Given any function $f: A \to B$ the function $\tilde{f}: A \to f(B)$ s.t. $\forall a \in A$, $\tilde{f}(a) = f(a)$ is a surjective function. But the functions $f, \tilde{f}$ are not equal since they have different codomains.

<u>Remark</u>:

$f: A \to B$ is surjective $\iff f(A) = B$.

<u>Definition 8</u>

The function f is a one-to-one correspondence or <u>bijection</u> if it is both injective and surjective.

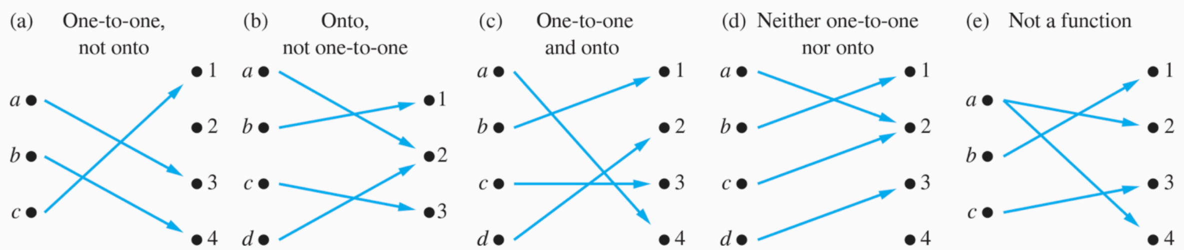


FIGURE 5 Examples of different types of correspondences.

Example:

Let A be a set. The identity function on A is the function $i_A : A \to A$, where $i_A(x) = x$ for all $x \in A$.

Since if $i_A(x) = i_A(y)$ then $x = y$ and if $x \in A$ (codomain) then $i_A(x) = x$ we have that i_A is bijective.

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Definition 10 Let A, B, and C be sets. Let f be a function from A to B and g be a function from B to C. The composition of the functions g and f denoted for all $a \in A$ by $g \circ f$, is the function from A to C defined by

$$(g \circ f)(a) = g(f(a)).$$

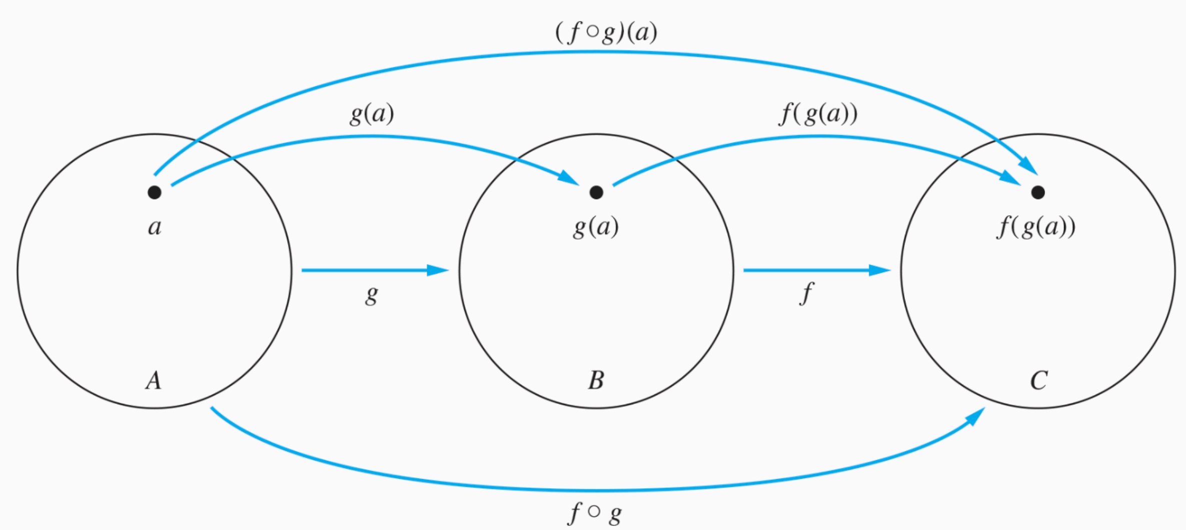


FIGURE 7 The composition of the functions f and g.

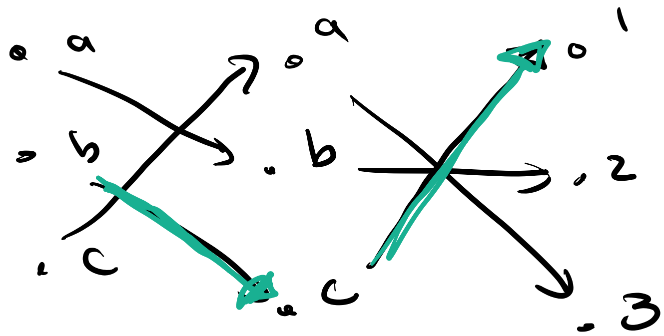
Example: Let $A = \{a, b, c\}$ and $f : A \to A$ s.t. $f(a) = b$, $f(b) = c$, $f(c) = a$. Let $B = \{1, 2, 3\}$ and $g : A \to B$ s.t. $g(a) = 3$, $g(b) = 2$, $g(c) = 1$. Then $(g \circ f) : A \to B$ is defined by

$$(g \circ f)(a) = g(f(a)) = g(b) = 2$$

$$(g \circ f)(b) = g(f(b)) = g(c) = 1$$

$$(g \circ f)(c) = g(f(c)) = g(a) = 3.$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "(g \circ f)"'] & A \arrow[r, "g"] & B \end{tikzcd}$$



$(g \circ f)(b)$

Example.

Let f and g be the functions from the set of integers to the set of integers defined by

$f(x) = 2x+3$ and $g(x) = 3x+2$.

$$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3$$
$$= 6x+7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2$$
$$= 6x+11.$$

Note: Usually $f \circ g$ and $g \circ f$ are different functions.

Example

Let f and g be the functions defined by

$f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ s.t. $f(x) = \sqrt{x}$ and

$g: \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ s.t. $g(x) = x^2$.

Then $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x.$

Note: This says that $(g \circ f)(x) = i_{\mathbb{R}^+ \cup \{0\}}(x)$.

Definition 9

A function f from A to B is invertible if there exists a function f^{-1} from A to B such that $f \circ f^{-1} = i_B$ and $f^{-1} \circ f = i_A$. The function f^{-1} is called the inverse of f.

Claim: f^{-1} is unique and f is invertible if and only if it is bijective.

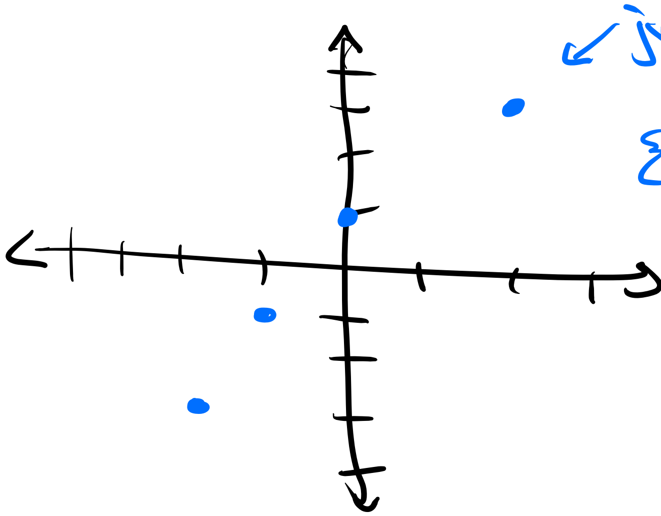
Definition 1

The graph of a function f from A to B is the set

$$\{(a, f(a)) \mid a \in A\}.$$

Example

Display the graph of the function $f(n) = 2n+1$ on the set of integers.



just discrete points since the graph is $\{(n, 2n+1) \mid n \in \mathbb{Z}\}$.

Remark: Graph of functions don't have to be continuous. (In fact they don't have to be points in $\mathbb{R} \times \mathbb{R}$).

Definition 12

The floor function assigns to a real number x the largest integer less than or equal to x. Denoted $\lfloor x \rfloor$. The ceiling function assigns to a real number x the smallest integer greater than or equal to x. Denoted $\lceil x \rceil$.

Remark: The floor function is often called the greatest integer function, denoted $[x]$.

Example:

$\lfloor 1/2 \rfloor = 0$, $\lceil 1/2 \rceil = 1$, $\lfloor -\frac{1}{2} \rfloor = -1$, $\lceil -1/2 \rceil = 0$

$\lfloor 7 \rfloor = \lceil 7 \rceil = 7$.

TABLE 1 Useful Properties of the Floor and Ceiling Functions.

(n is an integer, x is a real number)

(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
(1b) $\lceil x \rceil = n$ if and only if $n-1 < x \leq n$
(1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$
(1d) $\lceil x \rceil = n$ if and only if $x \leq n < x+1$
(2) $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$
(3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b) $\lceil x+n \rceil = \lceil x \rceil + n$

Example

Prove that if $x \in \mathbb{R}$ then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Suppose $x \in \mathbb{R}$. Then $x = n + \varepsilon$ where $n = \lfloor x \rfloor$ and $0 \le \varepsilon < 1$. This comes from the fact that $x - \lfloor x \rfloor < 1$ by 2 and k(a).

Since $x = n + \varepsilon$, $2x = 2n + 2\varepsilon$ where $2n \in \mathbb{Z}$ and $0 \le 2\varepsilon < 2$. We want two case if ① $\varepsilon \ge 1/2$ and ② $\varepsilon < 1/2$.

If $\varepsilon \ge 1/2$ then $2 > \varepsilon \ge 1$. So $\varepsilon = 1 + \varepsilon'$ where $0 \le \varepsilon' < 1$. Hence $\lfloor 2x \rfloor = \lfloor 2n + 1 + \varepsilon' \rfloor = 2n + 1$.

Now, $\lfloor x \rfloor = n$ and $\lfloor x + \frac{1}{2} \rfloor = \lfloor n + 1/2 + \varepsilon \rfloor$

However $1 > \varepsilon \ge 1/2$ so $2 > 1/2 + \varepsilon \ge 1$ so $1/2 + \varepsilon = 1 + \varepsilon''$ where $0 \le \varepsilon' < 1$. Hence $\lfloor n + 1/2 + \varepsilon \rfloor = \lfloor n + 1 + \varepsilon'' \rfloor = n + 1$.

$\therefore \lfloor 2x \rfloor = 2n + 1 = (n) + (n+1) = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$.

If $0 \le \varepsilon < 1/2$ then $0 \le 2\varepsilon < 1$ and $0 \le 1/2 + \varepsilon < 1$.

So, $\lfloor 2x \rfloor = \lfloor 2n + 2\varepsilon \rfloor = 2n$

$\lfloor x \rfloor = n$, and $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = n$.

$\therefore \lfloor 2x \rfloor = 2n = n + n = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$. ∎

<u>Remark:</u> Trick is to usually write $x = n + \varepsilon$ then break it into cases depending on the value of ε. For example something with $\lfloor 3x \rfloor$ we would have the cases

① $0 \le \varepsilon < 1/3$ ② $1/3 \le \varepsilon < 2/3$. ③ $2/3 \le \varepsilon < 1$.

<u>Definition</u>

The factorial function $f : \mathbb{N} \to \mathbb{Z}^+$ denoted by $f(n) = n!$ where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$. and $0! = 1$.