5.2 Strong Induction

Strong Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis Step:

We verify that $P(1)$ is true

Inductive Step:

We show the conditional statement

$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$.

However, we generally use a different format.

Basis Step:

We verify $P(b), \ldots, P(b+j)$ are true

Inductive Step:

We show that $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq b+j$.

Example

Show that if n is an integer greater than 1, then n can be written as a product of primes.

Proof:

Let $P(n)$ be that "n can be written as a product of primes"

Basis Case:

$P(2)$ is true, since 2 is prime.

Inductive Step:

Suppose $P(j)$ is true for $2 \le j \le k$, $j \in \mathbb{Z}$.

Two cases

(i) k+1 is prime:

If $k+1$ is prime then $k+1$ is a product of a prime (itself)

(ii) k+1 is composite:

If $k+1$ is composite then it is the product of two integers a, b where $2 \le a, b \le k$.

By the induction hypothesis a, b are the product of primes. Hence, $k+1 = ab$ is the product of primes. $\boxed{}$

Example:

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Basis Step:

12-cents is 3 4-cent stamps.

(Notice that if we have $P(k)$ then we get $P(k+4)$ since we can add a 4-cent stamp to k to get $k+4$ cents)

13-cents is 2 4-cent stamps and 1 5-cent stamps

14-cents is 1 4-cent stamp and 2 5-cent stamps.

15-cents is 3 - 5cent stamps.

Inductive Step:

Suppose $P(j)$ is true for $12 \le j < k$, where k is an integer ≥ 15.

Now since $k \ge 15$ we have that $k-3 \ge 12$. Hence, we can get $k-3$ cents from 4 and 5-cent stamps. If we add a 4-cent stamp then we can form $k+1$ from 4 and 5-cent stamps $\blacksquare$