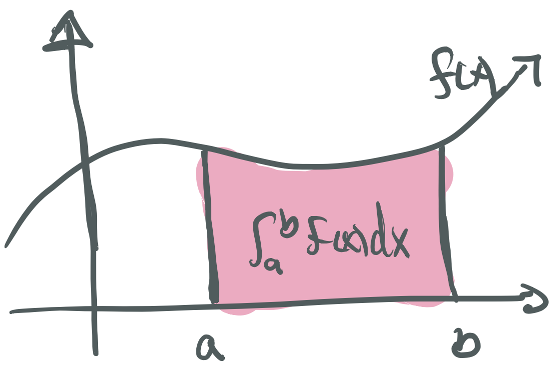
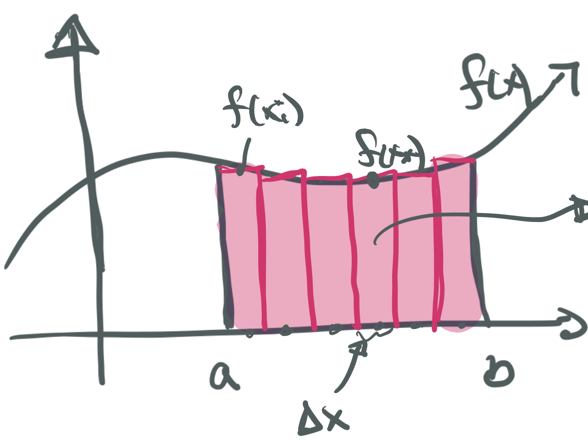
Recall from Calc I what an integral is.



We define this by looking at smaller and smaller rectangles



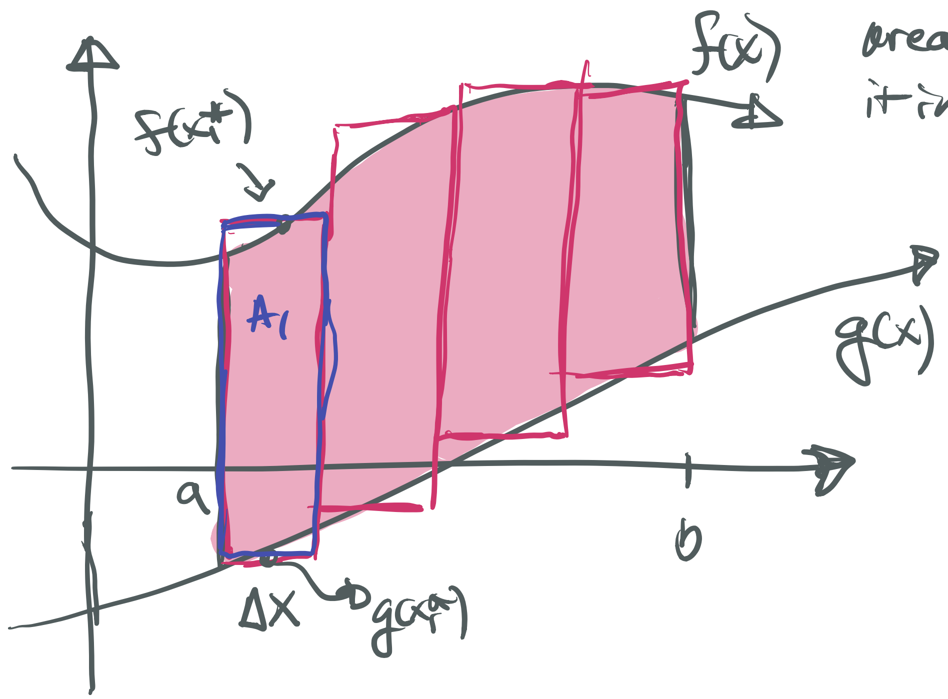
Area $= f(x_4^*)\Delta x$.

So the total area in this case is given by $\sum_{k=1}^{6} f(x_k^*)\Delta x$. The integral is the limit of taking smaller rectangles

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Now we want to extend this to region between curves

To find this area we divide it into rectangles



$$A_1 = \Delta x\,(f(x_i^*) - g(x_i^*)).$$

so the total area is roughly

$$\sum_{k=1}^{4} (f(x_i^*) - g(x_i^*))\Delta x.$$

We can take smaller and smaller rectangles to get a more exact area.

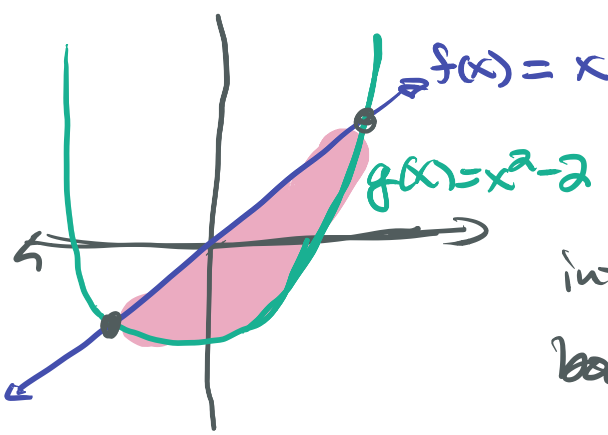
$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\Delta x$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(x_k^*)\Delta x\right) - \lim_{n\to\infty}\sum_{k=1}^{n} g(x_k^*)\Delta x$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Example:

Find the area between $f(x)=x$ and $g(x)=x^2-2$.



we need to find where the curves intersect to get the bounds. Set $f(x)=g(x)$.

$$x = x^2-2$$

$$0 = x^2-x-2 = (x-2)(x+1)$$

$$\Rightarrow \quad x=2 \quad \text{and} \quad x=-1.$$

$$\text{Area} = \int_{-1}^{2} f(x)-g(x)\,dx = \int_{-1}^{2} x-(x^2-2)\,dx$$

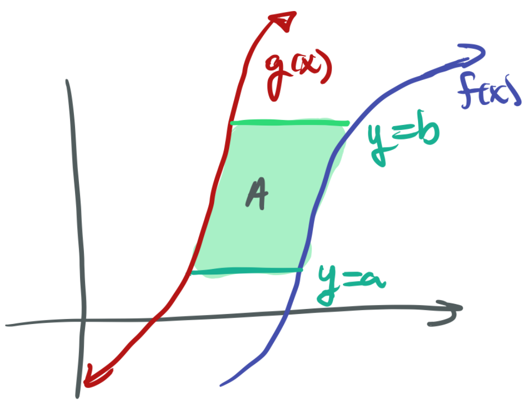
$$= \int_{-1}^{2} x-x^2+2\,dx = \frac{1}{2}x^2-\frac{1}{3}x^3+2x\Big|_{-1}^{2}$$

$$= \left[\tfrac{1}{2}(4) - \tfrac{1}{3}(8) + 2(2)\right] - \left[\tfrac{1}{2} + \tfrac{1}{3} - 2\right]$$

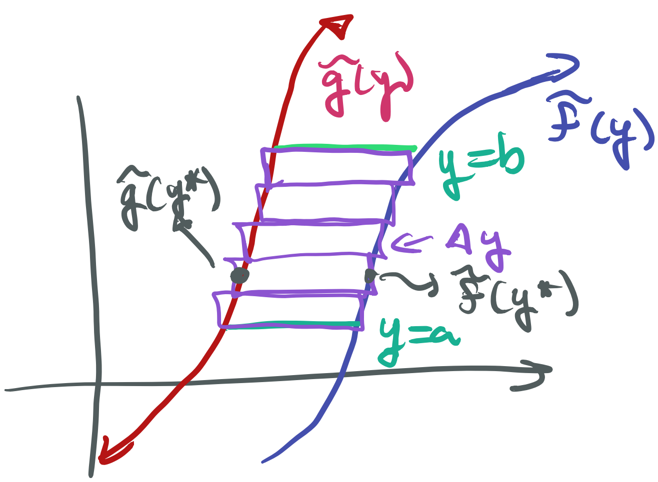
$$= 2 - 8/3 + 4 - \tfrac{1}{2} - \tfrac{1}{3} + 2$$

$$= 8 - \tfrac{9}{3} - \tfrac{1}{2} = \boxed{\tfrac{9}{2}}$$

What if we have two curves $f(x)$ and $g(x)$ and want to find the area of A?

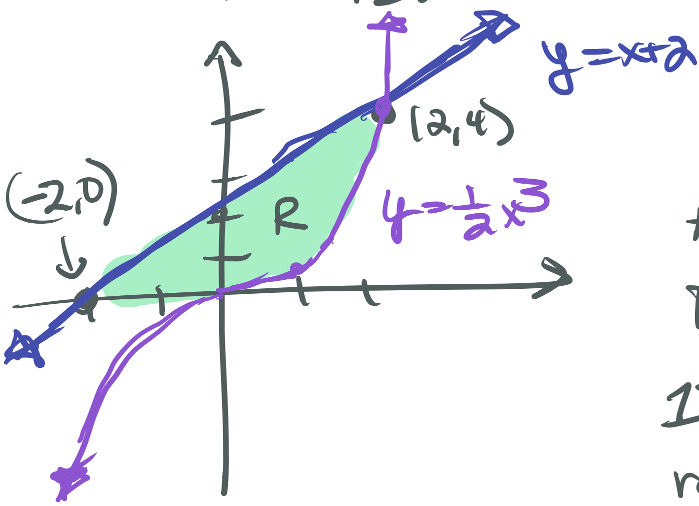


We can rewrite the functions in terms of y and do horizontal rectangles



$$\text{Area} = \int_a^b \tilde{g}(y) - \tilde{f}(y)\,dy$$

Example: Find the area of the region bounded by $y = x+2$, $y = \frac{1}{2}x^3$, and the x-axis.



There are two ways to approach this problem.

1) Integrate with respect to y.

2) Split into two pieces and integrate with respect to x.

Example:

1) y bounds are $y=0$ and $y=4$

$y = x+2 \Rightarrow x = y-2.$

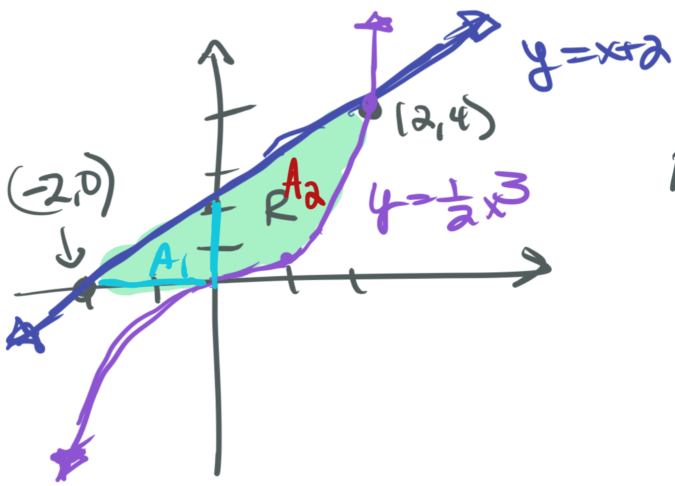
$y = \frac{1}{2}x^3 \Rightarrow (2y)^{1/3} = x.$

$$\text{Area} = \int_0^4 (2y)^{1/3} - (y-2)\,dy$$

$$= \frac{3}{8}(2y)^{4/3} - \left(\frac{1}{2}y^2 - 2y\right)\Big|_0^4$$

$$= \frac{3(8)^{4/3}}{8} - 8 + 8 = \boxed{6}$$

2)



$$A_1 = \int_{-2}^{0} x+2\,dx$$

$$= \frac{1}{2}x^2 + 2x\Big|_{-2}^{0}$$

$$= -(2-4) = 2.$$

$$A_2 = \int_0^2 (x+2) - \left(\frac{1}{2}x^3\right)dx$$

$$= \left[\frac{1}{2}x^2 + 2x - \frac{1}{8}x^4\right]_0^2$$

$$= 2 + 4 - \frac{16}{8} = 4$$

$$R = A_1 + A_2 = \boxed{6}$$

Note: Both methods agree.