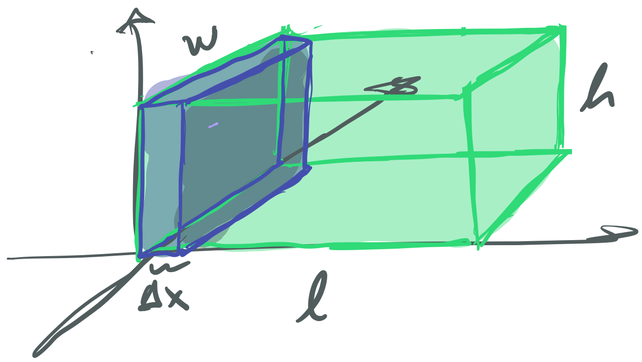
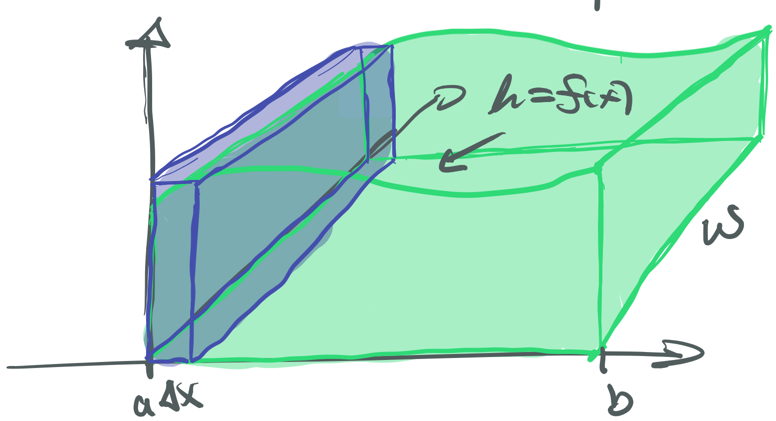
6.3 Volume by Slicing

Example: Let's start with a rectangular prism. The volume is given by $l \cdot w \cdot h$



but we can divide into smaller prisms and sum all the volumes. One of these smaller prisms has volume $w \cdot h \cdot \Delta x$. Note that $w \cdot h$ is the surface area of the gray face.

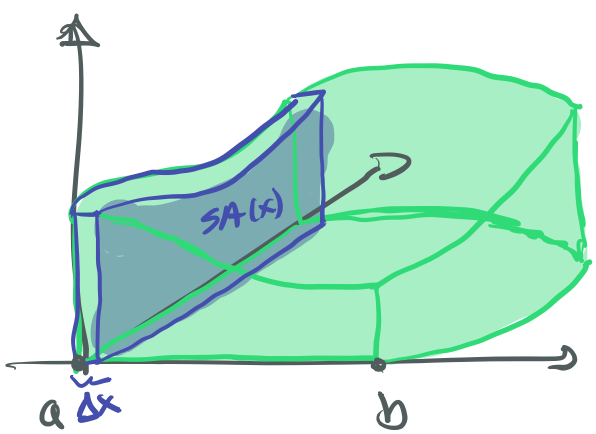
We can extend this idea to more complicated shapes. If we have constant width then the volume of a piece is given by $f(x) \cdot w \cdot \Delta x$.



Just like in two dimensions we can get the volume by taking smaller and smaller slices. So in this case,

$$\text{Volume} = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)\, w\, \Delta x = \int_a^b w \cdot f(x)\, dx.$$

Notice that $w \cdot f(x)$ is the surface area of the main face of each slice. In general if we can write the surface area of a slice in terms of x. Then,

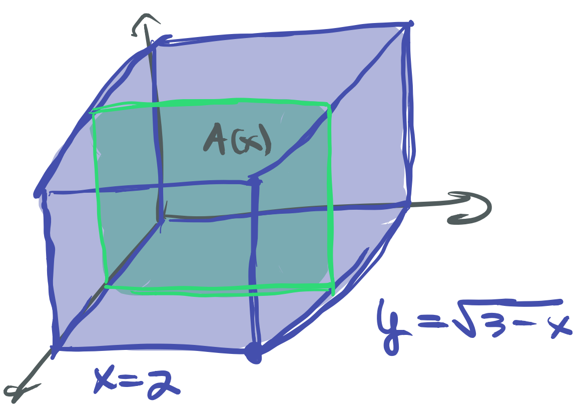


$$\text{Volume} = \int_a^b SA(x)\, dx$$

General Slicing Method

Suppose a solid object extends from $x=a$ to $x=b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a,b]$. The volume of the solid is $V = \int_a^b A(x)\,dx$.

Example: Consider a solid whose base is the region in the first quadrant bounded by the curve $y=\sqrt{3-x}$ and the line $x=2$ and whose cross sections through the solid perpendicular to the x-axis are squares. Find the volume of the solid.

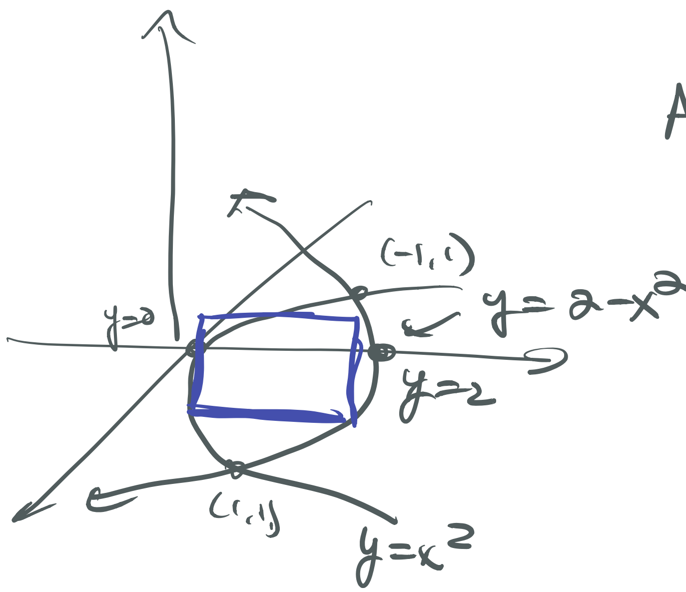


Each cross section is a square so

$$A(x) = \sqrt{3-x}\cdot\sqrt{3-x}$$
$$= 3-x$$

$$\text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 3-x\,dx$$
$$= \left[3x - \tfrac{1}{2}x^2\right]_0^2 = [6-2]-[0] = \boxed{4}$$

Example: Find the volume of the solid whose base is the region bounded by the curves $y = x^2$ and $y = 2 - x^2$, and whose cross sections are perpendicular to the x-axis are squares



$$A(x) = \left[(2-x^2) - (x^2)\right]^2$$

$$= \left[2 - 2x^2\right]^2$$

$$= 4 - 8x^2 + 4x^4$$

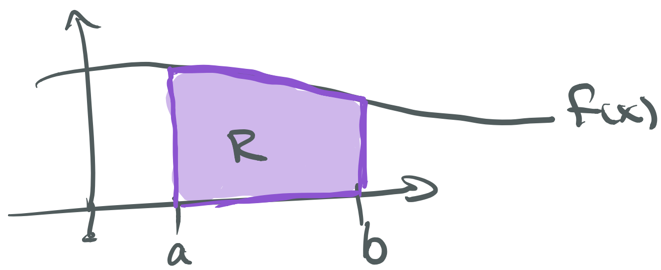
$$\text{Volume} = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} 2 - 8x^2 + 4x^4\,dx$$

$$= \left[4x - \frac{8}{3}x^3 + \frac{4}{5}x^5\right]_{-1}^{1} = \left[4 - \frac{8}{3} + \frac{4}{5}\right] - \left[-4 + \frac{8}{3} - \frac{4}{5}\right]$$

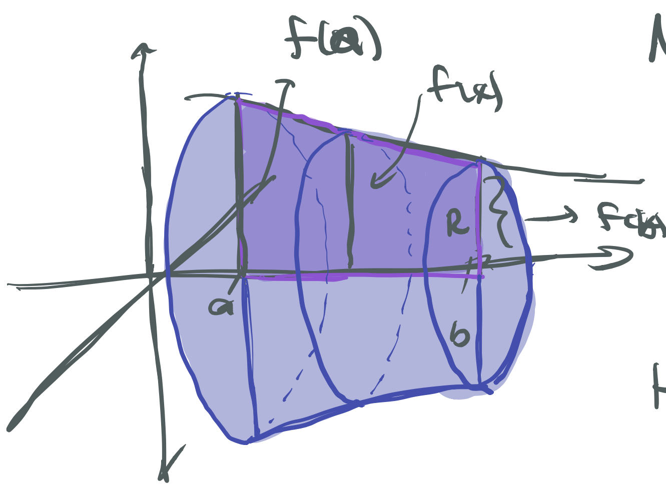
$$= 8 - \frac{16}{3} + \frac{8}{5} = \frac{120}{15} - \frac{80}{15} + \frac{24}{15} = \boxed{64/15}$$

Surface/solid of Revolution

Suppose f is a continuous function $f(x) \geq 0$ on a interval $[a,b]$. Let R be the region bounded by the graph of f, the x-axis and the lines $x=a$ and $x=b$.



Now we revolve R about the x-axis to give a three dimensional solid of revolution.



Notice that each cross section is a circle of radius $f(x)$.

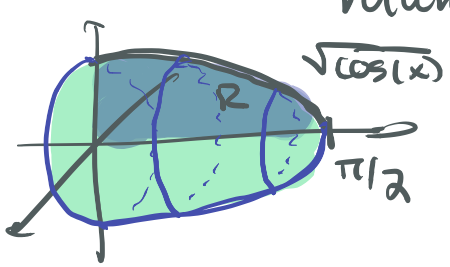
Hence $A(x) = \pi f(x)^2$

Disk Method about the x-axis

Let f be continuous with $f(x) \geq 0$ on the interval $[a,b]$. If the region R bounded by the graph of f, the x-axis and the lines $x=a$ and $x=b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi f(x)^2 \, dx$$

Example: Let R be the region bounded by the curve $y=\sqrt{\cos x}$ and the x-axis on $[0, \pi/2]$. A solid of revolution is obtained by revolving R about the x-axis. Find the volume of the solid.



$$V = \int_0^{\pi/2} \pi f(x)^2 \, dx$$

$$= \int_0^{\pi/2} \pi \left(\sqrt{\cos(x)}\right)^2 dx$$

$$= \int_0^{\pi/2} \pi \cos(x) \, dx$$

$$= \pi \sin(x) \Big|_0^{\pi/2}$$

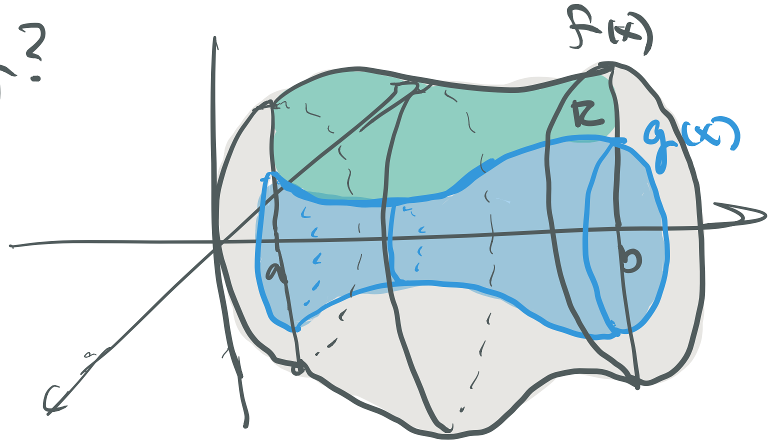
$$= \pi - 0 = \boxed{\pi}$$

Washer Method

This is just an extension of the disk method. Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a,b]$. Let R be the region bounded by $y=f(x)$, $y=g(x)$ and the lines $x=a$, $x=b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$\boxed{V = \int_a^b \pi \left(f(x)^2 - g(x)^2 \right) dx}$$

Why?



we can see that the volume of R after it's revolved is the volume of the Gray figure minus the volume of the blue.

$$\text{Volume Gray} = \int_a^b \pi f(x)^2 dx$$

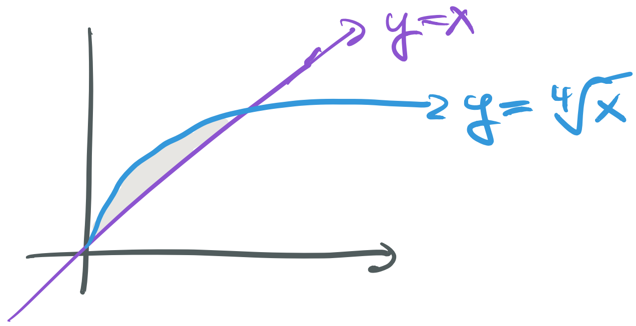
$$\text{Volume Blue} = \int_a^b \pi g(x)^2 dx$$

so the volume we want is

$$V = \int_a^b \pi f(x)^2 dx - \int_a^b \pi g(x)^2 dx$$

$$= \int_a^b \pi \left[f(x)^2 - g(x)^2 \right] dx.$$

Example: Find the volume of the region bounded by $y=x$ and $y=\sqrt[4]{x}$ about the x-axis.



We need to find when $x=\sqrt[4]{x}$. This only happens when $x=0$ or $x=1$ so the bounds on the integral are 0 and 1

$$\text{Volume} = \int_0^1 \pi\left((\sqrt[4]{x})^2 - x^2\right) dx$$

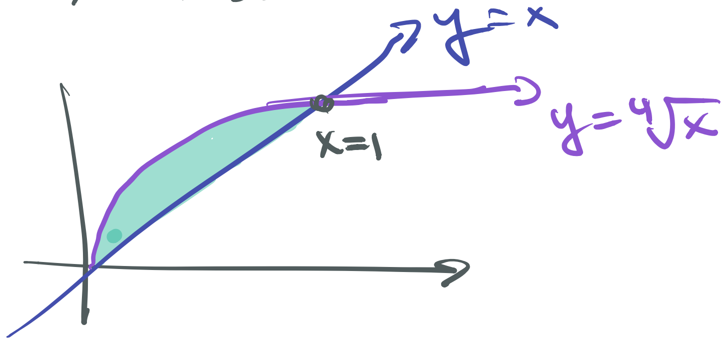
$$= \int_0^1 \pi\left[x^{1/2} - x^2\right] dx$$

$$= \pi\left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$$

$$= \boxed{\frac{\pi}{3}}$$

Example:

Find the volume of region bounded by $y=x$ and $y=\sqrt[4]{x}$ rotated about the x-axis.

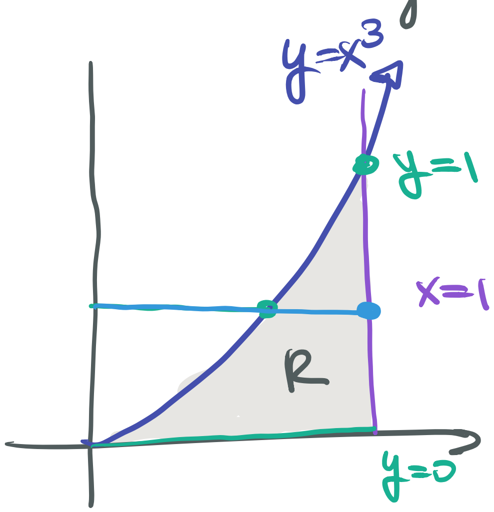


$$\text{Volume} = \int_0^1 \pi\left((\overset{x^{1/2}}{\overline{x^{1/4})^2}} - x^2\right)dx$$

$$= \frac{\pi 2}{3}x^{3/2} - \frac{\pi}{3}x^2\Big|_0^1 = \boxed{\frac{\pi}{3}}$$

We can rotate functions about other lines. We just need to find the radius. We can also rotate about the y-axis but then we need functions of the form $x=f(y)$.

Example: Find the volume of the region bounded by $y=x^3$, $y=0$ and $x=1$ rotatated about the y-axis.



The outer radius is 1 the inner radius is $y=x^3$ but we need $x=f(y)$ so we rewrite this as $x=\sqrt[3]{y}$. The bounds are $y=0$ and $y=1$.

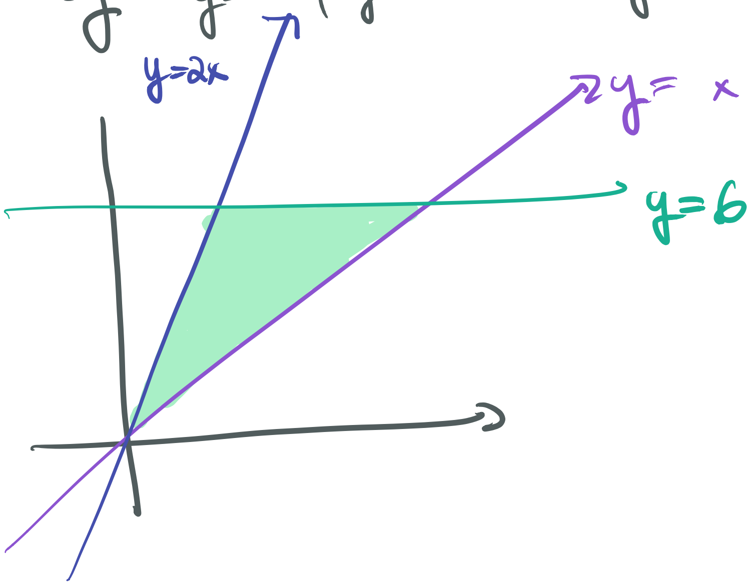
$$\text{Volume} = \pi\int_0^1 1^2-(y^{1/3})^2\,dy=$$

$$= \pi\left[y - \frac{3}{5}y^{5/3}\right]_0^1$$

$$= \boxed{\frac{2\pi}{5}}$$

Example:

Find the volume of the region bounded by $y=x$, $y=2x$ and $y=6$ rotated about the y-axis



$y=2x$

$$\Rightarrow x = \frac{y}{2}$$

$$\text{Volume} = \int_0^6 \pi\left(y^2 - \left(\frac{y}{2}\right)^2\right)dy$$

$$= \pi\left[\frac{y^3}{3} - \frac{y^3}{12}\right]_0^6$$

$$= \frac{\pi}{3}\left[y^3 - \frac{y^3}{4} \right]_0^6$$

$$= \frac{\pi}{3}\left[\frac{3y^3}{4} \right]_0^6 = \frac{216\pi}{4} = 54\pi$$