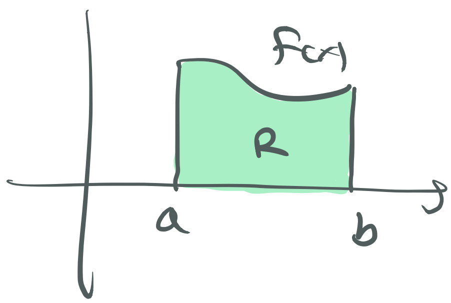
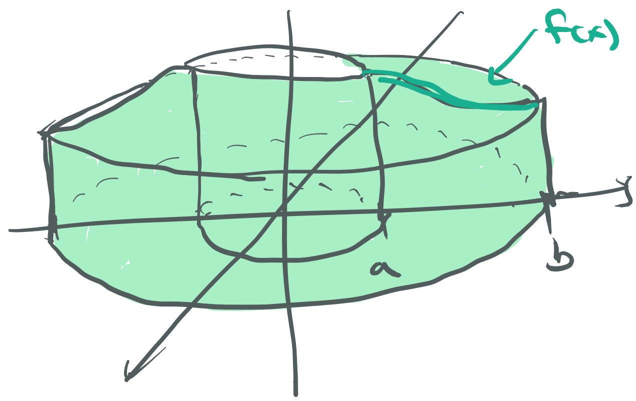
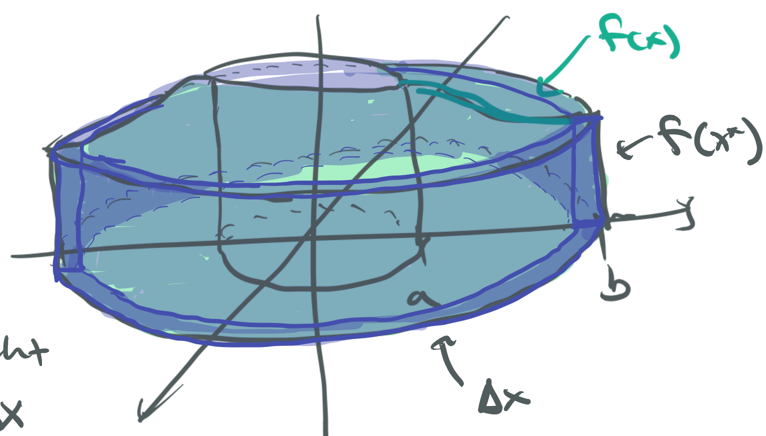
6.4 Volumes by Shells

Consider the region bounded by $f(x) \geq 0$ the x-axis and the lines $x=a$ and $x=b$. We want to find the volume of the region rotated about the y-axis.





Now we can divide this into shells and sum to get the volume. The volume of the shell is roughly $\underbrace{2\pi x}_{\text{circumference}}\,\overbrace{f(x)}^{\text{height}}\,\underbrace{\Delta x}_{\text{width.}}$



So

$$\text{Volume} = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_k f(x_k^*)\,\Delta x = \int_a^b 2\pi x f(x)\,dx.$$

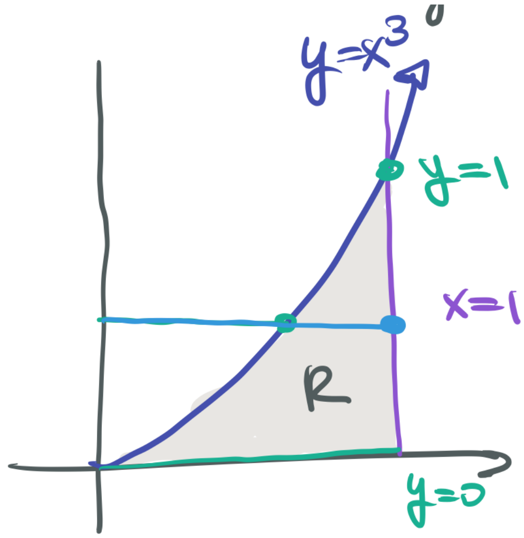
We can generalize this to situations where the lower y-bound on the region is a function $g(x)$.

<u>Volume by shell method</u>

Let $f(x)$ and $g(x)$ be continuous functions with $f(x) \geq g(x)$ on $[a,b]$. If R is the region bounded by the curves $y=f(x)$ and $y=g(x)$ between the lines $x=a$ and $x=b$, the volume of the solid generated when R is revolved about the y-axis is

$$V = \int_a^b 2\pi x\,(f(x)-g(x))\,dx.$$

<u>Example</u>: Find the volume of the region bounded by $y=x^3$, $y=0$ and $x=1$ rotatated about the y-axis.



We originally did this problem by switching to a y-integral but we can use the shell method instead.

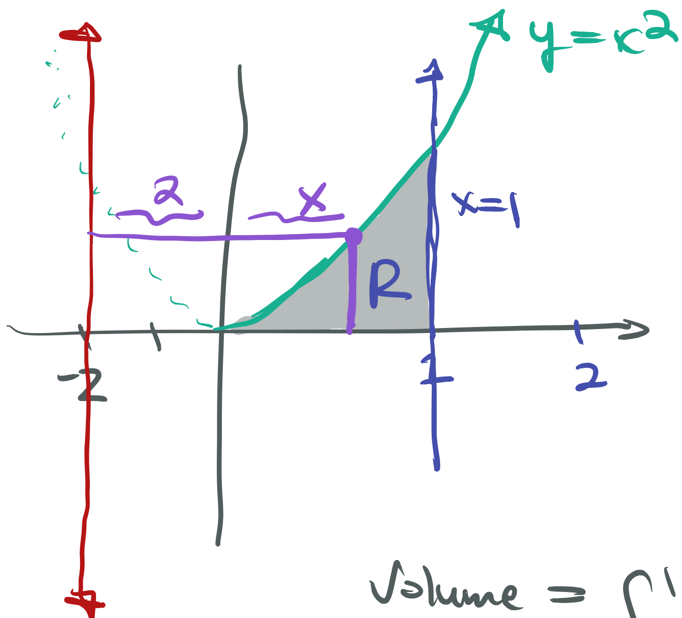
$$\text{Volume} = \int_0^1 2\pi x\,(x^3)\,dx$$

$$= 2\pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{2\pi}{5}$$

Which agrees with the volume we found using the washer method.

Example: Let R be the region bounded by $y=x^2$, $x=1$ and $y=0$. Find the volume using the shell method

When R is rotated about the line $x = -2$



we see from the picture that the radius is $x+2$ and the height is x^2 and the bounds are $x=0$ to $x=1$

$$\text{Volume} = \int_0^1 2\pi (x+2) \cdot x^2 \, dx$$

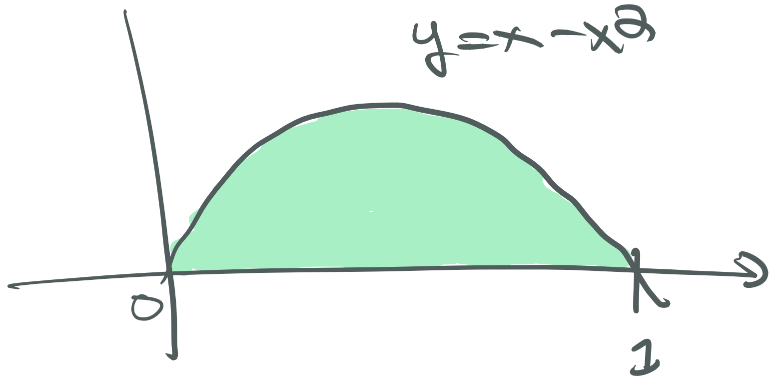
$$= 2\pi \int_0^1 x^3 + 2x^2 \, dx$$

$$= 2\pi \left[\frac{1}{4}x^4 + \frac{2}{3}x^3 \right]_0^1 =$$

$$= 2\pi \left[\frac{1}{4} + \frac{2}{3} \right] = 2\pi \left[\frac{3}{12} + \frac{8}{12} \right]$$

$$= \boxed{\frac{11\pi}{6}}$$

Example: Find the volume of the region bounded by $y = x - x^2$, $y = 0$ rotated about the y-axis.



$$\text{volume} = 2\pi \int_0^1 x\,(x - x^2)\,dx$$

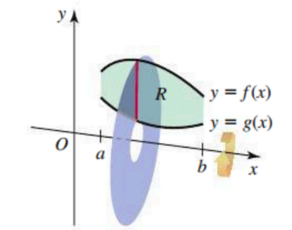
$$= 2\pi \int_0^1 x^2 - x^3\,dx$$

$$= 2\pi \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 \right]_0^1$$

$$= 2\pi \left[\tfrac{1}{3} - \tfrac{1}{4} \right] = \boxed{\frac{\pi}{6}}$$

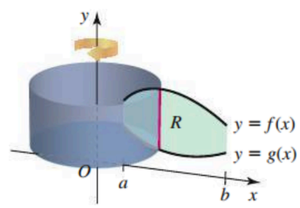
SUMMARY Disk/Washer and Shell Methods

Integration with respect to x



Disk/washer method about the x-axis
Disks/washers are *perpendicular* to the x-axis.

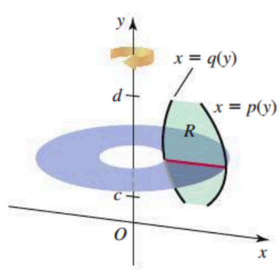
$$\int_a^b \pi(\underbrace{f(x)}_{\text{outer radius}}{}^2 - \underbrace{g(x)}_{\text{inner radius}}{}^2)\,dx$$



Shell method about the y-axis
Shells are *parallel* to the y-axis.

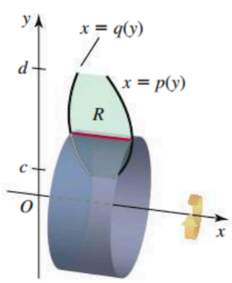
$$\int_a^b \underbrace{2\pi x}_{\text{shell circumference}}\underbrace{(f(x) - g(x))}_{\text{shell height}}\,dx$$

Integration with respect to y



Disk/washer method about the y-axis
Disks/washers are *perpendicular* to the y-axis.

$$\int_c^d \pi(\underbrace{p(y)}_{\text{outer radius}}{}^2 - \underbrace{q(y)}_{\text{inner radius}}{}^2)\,dy$$



Shell method about the x-axis
Shells are *parallel* to the x-axis.

$$\int_c^d \underbrace{2\pi y}_{\text{shell circumference}}\underbrace{(p(y) - q(y))}_{\text{shell height}}\,dy$$