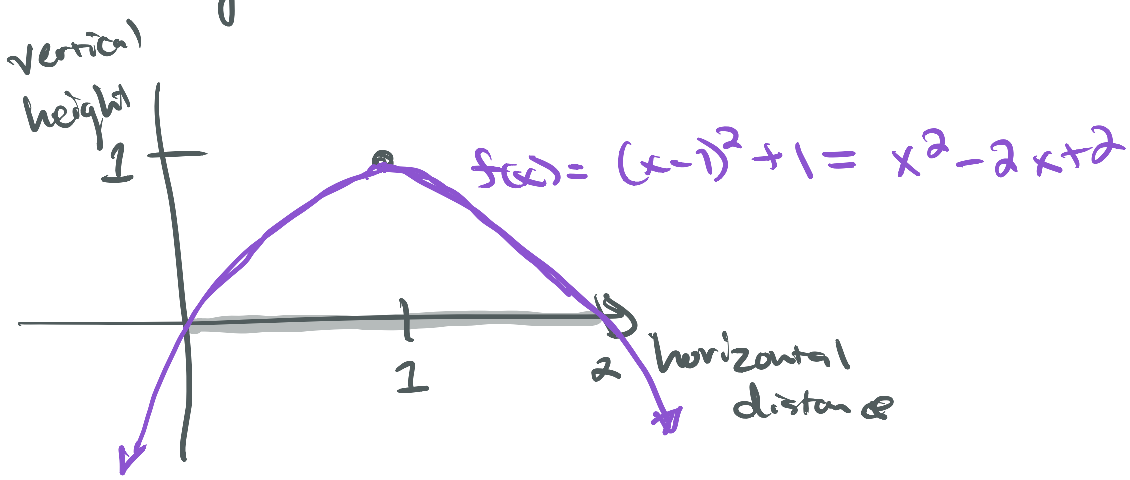
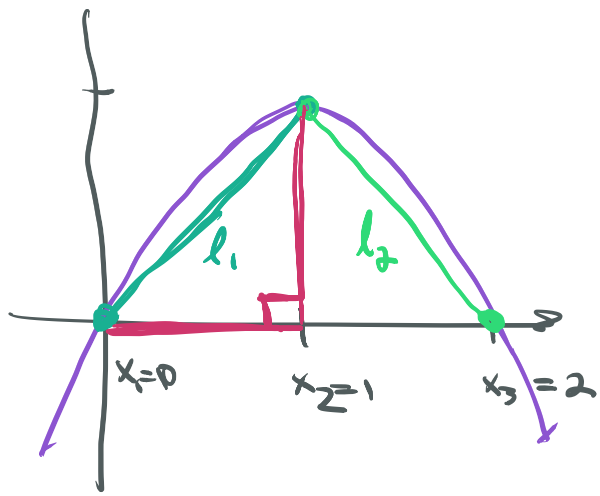
6.5 Length of Curves

Ryan Crouser just set the shot put world record at 23.37 meters, but this only tracks the horizontal distance the shot put. How could we go about finding the total distance traveled? Let's start by using a simplified example. A projectile without air resistance follows a parabolic path so let's look at $f(x) = (x-1)^2 + 1$ and try to find the total distance.



If we were to just track horizontal distance we would get 2 but this misses all of the vertical distance. We can try to approximate the distance by finding the length of l_1 and l_2. These are just lines



so we can find the length using the Pythagorean theorem.

$$\text{Length } l_1 = \sqrt{(x_2-x_1)^2 + (f(x_2)-f(x_1))^2}$$

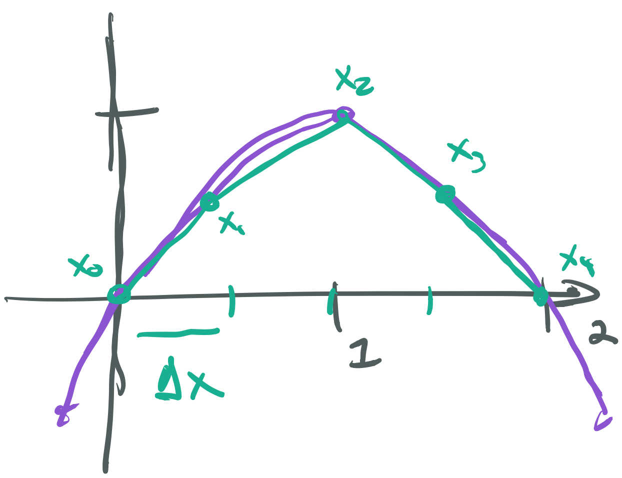
$$= \sqrt{1^2+1^2} = \sqrt{2}$$

$$\text{length } l_2 = \sqrt{(x_3-x_2)^2 + (f(x_3)-f(x_2))^2} = \sqrt{2}$$

So the length of the curve is at least

$2\sqrt{2} \approx 2.8$ which is significantly different than just the horizontal distance.

Now we can take more and more pieces to get a better approximation.



If we use 4 lines with the Δx the same we get:

$$\text{length} = \sum_{k=0}^{3} \sqrt{(\Delta x)^2 + (f(x_{k+1}) - f(x_k))^2}$$

Since $x_{k+1} = x_k + \Delta x$ we can rewrite this as

$$\text{length} = \sum_{k=0}^{3} \sqrt{(\Delta x)^2 + (f(x_k + \Delta x) - f(x_k))^2}$$

$$= \sum_{k=0}^{3} (\Delta x) \sqrt{1 + \left(\frac{f(x_k + \Delta x) - f(x_k)}{\Delta x}\right)^2}$$

Now we want to find the exact length so we take $\Delta x \to 0$. Hence

$$\text{length} = \lim_{n \to \infty} \sum_{k=0}^{n} (\Delta x) \sqrt{1 + \left(\frac{f(x_k + \Delta x) - f(x_k)}{\Delta x}\right)^2}$$

$$\lim_{\Delta x \to 0} \frac{f(x_k + \Delta x) - f(x_k)}{\Delta x} = f'(x_k)$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n} (\Delta x) \sqrt{1 + f'(x_k^*)^2}$$

$$= \int_0^2 \sqrt{1 + f'(x)^2}\, dx$$

$f(x) = (x-1)^2 + 1 \Rightarrow f'(x) = 2(x-1)$

$$= \int_0^2 \sqrt{1 + 4(x-1)^2}\, dx \approx 2.96$$

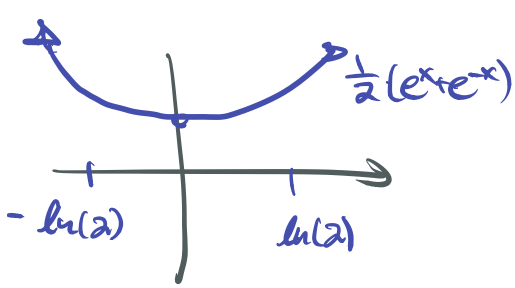
Remark: Frequently these will be hard integrals and the set-up is more important than the answer.

Definition Arclength for $y=f(x)$

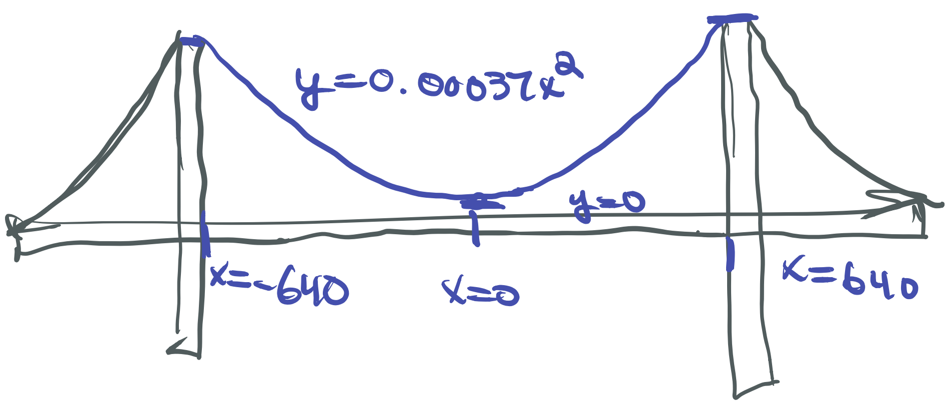
Let f be a continuous first derivative on the interval $[a,b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$\boxed{L = \int_a^b \sqrt{1+f'(x)^2}\,dx}$$

Example: Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ on the interval $[-\ln(2), \ln(2)]$.



Example: The cables holding up suspension bridges are roughly parabolic. Arclength can help find the right length of cable needed for the bridge. The central span of the Golden Gate Bridge is 1280m long and 152m high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \leq 640$ and x and y are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.



Example: Find the arclength of the curve $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on $[1, 2]$