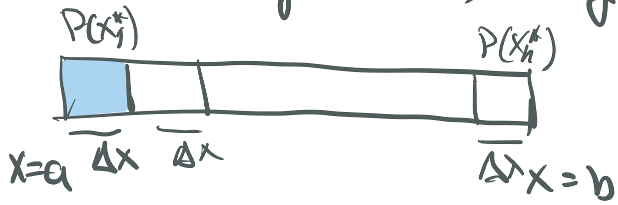
6.7 Physical Applications

Density and Mass

An object with uniform density satisfies mass = density $\cdot$ volume. However and object might not have uniform density.

Suppose we have a thin rod with density $p(x)$ which varies. Then we can approximate the density by dividing the rod into small pieces

density is $P(x)$ in g/m.



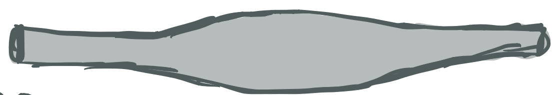
The total mass is then approximately $\sum_{k=1}^{n} P(x_k^*)\Delta x$. Taking Δx we have

$$\text{mass} = \lim_{n\to\infty} \sum_{k=1}^{n} P(x_k^*)\Delta x = \int_a^b p(x)\,dx.$$

Example: Find the mass of the thin bar with the density function

$p(x) = 1+\sin(x)$ for $0 \le x \le \pi$.

Note: This could represent a bar with a thicker center and thinner ends



$x=0$ $x=\pi$

$$\text{mass} = \int_0^{\pi} 1+\sin(x)\,dx = \left[x-\cos(x)\right]_0^{\pi}$$

$$= (\pi+1)-(-1) = \boxed{\pi+2}$$

Example: Find the mass of the thin bar with density $p(x) = 5e^{-2x}$ for $0 \le x \le 4$.

$$\text{mass} = \int_0^4 p(x)\,dx$$

$$= \int_0^4 5e^{-2x}\,dx$$

$$= \left[-\frac{5}{2}e^{-2x}\right]_0^4$$

$$= -\frac{5}{2}e^{-8} + \frac{5}{2}e^{-2\cdot 0}$$

$$= \boxed{\frac{5}{2} - \frac{5}{2}e^{-8}}$$

<u>Work</u>: work = Force · distance

So if we have a variable force given by $F(x)$ we can estimate the work done from $x=a$ to $x=b$ by dividing it into small pieces and summing them so $\text{Work} \approx \sum_{k=1}^{n} F(x_k^*)\Delta x$

Once again taking $\lim_{n\to\infty}$ we get

$$\text{Work} = \int_a^b F(x)\,dx.$$

Example: Hooke's Law.

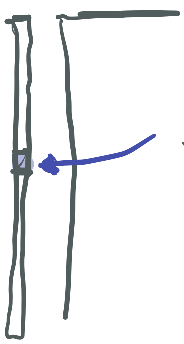
The force required to keep a spring compressed or stretched x units from equilibrium is $F(x) = kx$.

Find the work required to stretch a spring by 2 units

$$W = \int_0^2 kx\,dx = \frac{k}{2}x^2\Big|_0^2 = \boxed{2k}$$

Example: Lifting problems

The gravitational force on an object with mass m (in kg) is $F = mg$ where $g = 9.8\,m/s^2$. Suppose we want to lift a chain with density $\rho(y)$.



this piece has mass $\rho(y_k^*)\Delta y$

so the force is $g\,\rho(y_k^*)\Delta y$

we have to move it a distance of $h-y_k^*$

so $\text{work} = \lim_{n\to\infty}\sum_{k=1}^{n} g\,\rho(y_k^*)(h-y_k^*)\Delta y$

$$= \int_0^h g\,\rho(y)(h-y)\,dy.$$

A 30m long chain hangs vertically from a winch. Assume there is no friction and the density is 5kg/m. How much work is required to wind the entire chain?

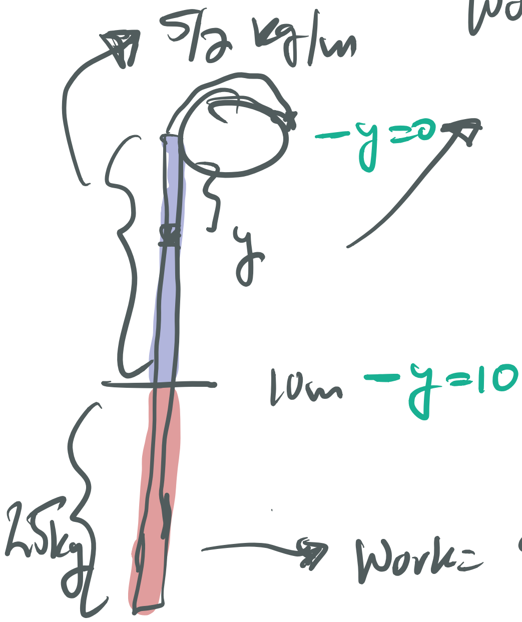
$$\text{Work} = \int_0^{30} 9.8\cdot 5(30-y)\,dy$$

$$= 9.8\cdot 5\left(30y - y^2/2\right)\Big|_0^{30}$$

$= 9.8 \cdot 5\,(900 - 900/2)$

$= 4.9 \cdot 5\,(900/2) = \boxed{11025\text{ J}}$

Example: A 20m long 50-kg chain hangs vertically from a cylinder attached to a winch. How much work is required to wind the upper half of the chain onto the winch?



$$\text{Work} = \int_0^{10} 9.8 \cdot \frac{5}{2} \cdot y\,dy$$

$$= \left[\frac{5}{4} \cdot 9.8 \cdot y^2\right]_0^{10}$$

$$= \frac{500}{4} \cdot 9.8 = 125 \cdot 9.8\text{ J}$$

$\text{Work} = 9.8\text{ m/s}^2 \cdot 25\text{ kg} \cdot 10\text{ m} = 250 \cdot 9.8\text{ J}$

$\text{Total Work} = 375 \cdot 9.8 = \boxed{3675\text{ J}}$

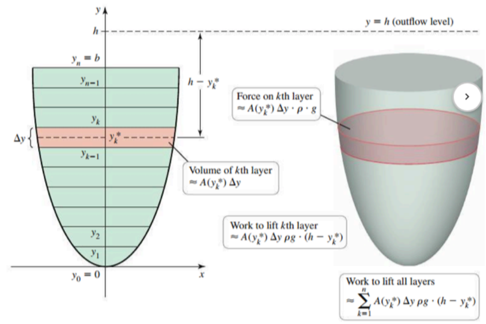


Figure 6.76

Example: Pumping Water

PROCEDURE **Solving Pumping Problems**

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.
2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.
3. The work required to lift the water is

$$W = \int_a^b \rho g A(y) D(y)\, dy.$$

A cylindrical tank has height 8m and radius 2m. If the tank is full how much work is required to pump the water to the level out of the top of the tank and out of the tank?

Density of water is 1000kg/m^3.

$A(y) = \pi r^2 = 4\pi \cdot m^2 \quad g = 9.8 m/s^2$

$D(y) = (8-y) m. \quad \rho = 1000 kg/m^3$

$$\text{Work} = \int_0^8 4\pi(9.8)(1000)(8-y)\,dy$$

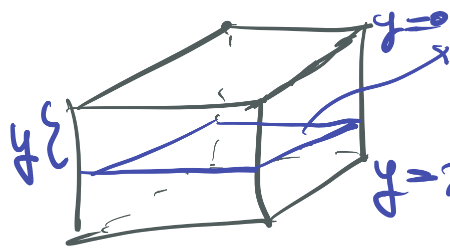
$$= 4\pi(9.8)(1000)\int_0^8 (8-y)\,dy$$

$$= (4000\pi)(9.8)\left[8y - y^2/2\right]_0^8$$

$$= 4000\pi(9.8)\left[64 - 64/2\right]$$

$$= \boxed{1254400\pi J}$$

Example: A swimming pool has the shape of a box with a base that measures 25m × 15m and a uniform depth of 2.5m. How much work is required to pump the water out of the pool?



Area = 25 · 15 = 375

$$\text{Work} = \int_0^{2.5} \overset{g}{9.8} \cdot \overset{\rho}{1000} \cdot \overset{A(y)}{375} \cdot \overset{D(y)}{y}\, dy$$

$$= 3675000 \int_0^{2.5} y\, dy$$

$$= 367500 \left[\tfrac{1}{2} y^2\right]_0^{2.5}$$

$$= \boxed{11\,484\,375 \text{ J}}$$

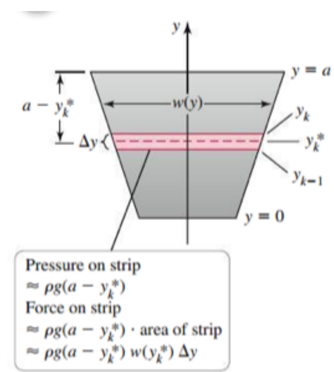


Figure 6.79

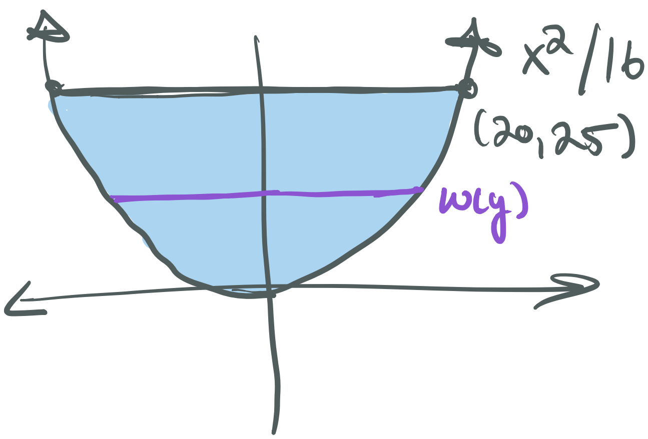
Example Force-on-dam Problems

PROCEDURE Solving Force-on-Dam Problems

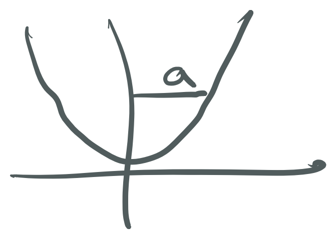
1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).
2. Find the width function $w(y)$ for each value of y on the face of the dam.
3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho g \underbrace{(a - y)}_{\text{depth}} \underbrace{w(y)}_{\text{width}}\, dy.$$

The lower edge of a dam is defined by the parabola $y = x^2/16$ with a height of 25 meters. Find the total force on the dam.



Note: The dam is symmetric so if we find



then $w(y) = 2a$. $y = x^2/16$

so $x = \pm\sqrt{16y} = \pm 4\sqrt{y}$

a is the positive part. So

$w(y) = 8\sqrt{y}$. depth $= 25 - y$

$$F = \int_0^{25} 1000(9.8)\, 8\sqrt{y}\,(25-y)\,dy$$

$$= 9800 \int_0^{25} 200y^{1/2} - 8y^{3/2} dy$$

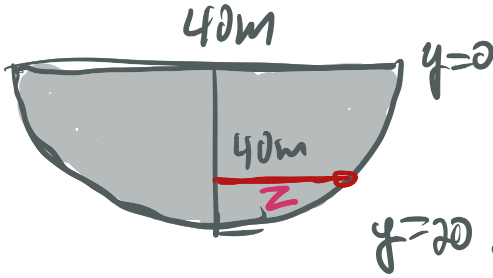
$$= 9800 \left[\frac{400}{3} y^{3/2} - \frac{16}{5} y^{5/2} \right]_0^{25}$$

$$= 9800 \left[\frac{400}{3}(125) - 16(625) \right]$$

$$= \boxed{65333333}$$

Example:

Find the force on a dam with the shape of a semicircle with diameter 40m.



Recall the equation for a circle is

$$x^2 + y^2 = 400$$

Solving for x we get $x = \sqrt{400-y^2}$
gives z. so $w(y) = 2\sqrt{400-y^2}$

$$\text{Force} = \int_0 \rho g \, D(y) \, w(y) \, dy$$

$$= \int_0^{40} 9.8 \cdot 1000 \cdot y \cdot 2\sqrt{400-y^2} \, dy$$

$$= 19600 \int_0^{40} y\sqrt{400-y^2} \, dy$$

$$= 19600 \left[\frac{1}{3} (400-y^2)^{3/2} \right]_0^{20}$$

$$= \frac{19600}{3} [400]^{3/2}$$

$$= \frac{19600}{3} \cdot 20^3 = \boxed{156800000/3 \text{ N}}$$