8.1 Basic approaches

Table 8.1 Basic Integration Formulas

1. $\int k\,dx = kx + C$, k real	2. $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C,\ p \neq -1$ real	3. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
4. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$	5. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$	6. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
7. $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$	8. $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$	9. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
10. $\int \frac{dx}{x} = \ln\lvert x\rvert + C$	11. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$	12. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C,\ a > 0$
13. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C,\ a > 0$	14. $\int \tan ax\,dx = \frac{1}{a}\ln\lvert\sec ax\rvert + C$	15. $\int \cot ax\,dx = \frac{1}{a}\ln\lvert\sin ax\rvert + C$
16. $\int \sec ax\,dx = \frac{1}{a}\ln\lvert\sec ax + \tan ax\rvert + C$	17. $\int \csc ax\,dx = -\frac{1}{a}\ln\lvert\csc ax + \cot ax\rvert + C$	

Example Substitution Show formula 15

$$\int \cot(ax)\,dx = \int \frac{\cos(ax)}{\sin(ax)}\,dx$$

$u = \sin(ax)$

$du = a\cos(ax)\,dx$

$$= \frac{1}{a}\int \frac{1}{u}\,du$$

$$= \frac{1}{a}\ln|u| + C$$

$$= \frac{1}{a}\ln|\sin(ax)| + C$$

Example: Evaluate $\int \cot x \csc^2 x \, dx$

$u = \csc(x)$

$du = -\cot(x)\csc(x)\, dx$

$$\int \cot x \csc^2 x \, dx = -1 \int \csc(x) \left[-\cot x \, cs(x) \right] dx$$

$$= -\int u \, du$$

$$= -\frac{1}{2} u^2 + c$$

$$= \boxed{\frac{-1}{2} \csc^2 x + C}$$

Example *Multiplication by 1* Show formula 17

$$\int \csc(ax)\,dx = \int \csc(ax)\left[\frac{\csc(ax)+\cot(ax)}{\csc(ax)+\cot(ax)}\right]dx$$

$$= \int \frac{\csc^2(ax) + \csc(ax)\cot(ax)}{\csc(ax) + \cot(ax)}\,dx$$

$u = \csc(ax) + \cot(ax)$

$du = -a\csc(ax)\cot(ax) + -a\csc^2(ax)$

$$= \frac{-1}{a}\int \frac{1}{u}\,du$$

$$= \frac{-1}{a}\ln|u| + C$$

$$= \frac{-1}{a}\ln\left|\csc(ax) + \cot(ax)\right| + C.$$

Example Evaluate $\int \frac{d\theta}{1+\sin\theta}$

Hint: Use $\sin^2\theta + \cos^2\theta = 1$

$$\int \frac{d\theta}{1+\sin\theta} = \int \frac{1}{1+\sin\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta}\, d\theta$$

$$= \int \frac{1-\sin\theta}{1-\sin^2\theta}\, d\theta$$

$$= \int \frac{1-\sin\theta}{\cos^2\theta}\, d\theta$$

$$= \int \sec^2\theta - \sec\theta\tan\theta\, d\theta$$

$$= \tan\theta - \sec\theta + C$$

Example Subtle Substitution

Evaluate $\int \frac{e^{2z}}{e^{2z}-4e^{-z}}\,dz$

$$\int \frac{e^{2z}}{e^{2z}-4e^{-z}}\,dz = \int \frac{e^{3z}}{e^{3z}-4}\,dz$$

$u = e^{3z}-4$

$du = 3e^{3z}$

$$= \frac{1}{3}\int \frac{1}{u}\,dz$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|e^{3z}-4| + C$$

Example: Evaluate $\int \frac{e^{2x}}{e^{4x}+1}\,dx$

$u = e^{2x}$

$du = 2e^{2x}\,dx$

$$\int \frac{e^{2x}}{e^{4x}+1}\,dx = \int \frac{\frac{1}{2}du}{u^2+1}$$

$$= \frac{1}{2}\arctan(u) + C$$

$$= \boxed{\frac{1}{2}\arctan(e^{2x}) + C}$$

Example Split up fractions

Evaluate $\int \frac{\sin x + 1}{\cos x}\,dx$

$$\int \frac{\sin x+1}{\cos x}\,dx = \int \frac{\sin x}{\cos x} + \sec x\,dx$$

This is #16

$$= \int \frac{\sin x}{\cos x}\,dx + \int \sec x\,dx$$

16. $\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\int \frac{1}{u}\,du + \int \sec x\,dx$$

$$= -\ln|u| + \ln|\sec x + \tan x| + C$$

$$= -\ln|\cos x| + \ln|\sec x + \tan x| + C$$

$$= \ln\left|\frac{1}{\cos x}(\sec x + \tan x)\right| + C$$

$$= \ln|\sec^2 x + \sec x \tan x| + C$$

$$= \boxed{\ln\left|\frac{1+\sin x}{\cos 2x}\right| + C}$$

Example Evaluate $\displaystyle\int_0^{\pi/4} \frac{\sec\theta+\csc\theta}{\sec\theta\csc\theta}\,d\theta$

$$\int_0^{\pi/4} \frac{\sec\theta+\csc\theta}{\sec\theta\csc\theta}\,d\theta = \int_0^{\pi/4} \frac{1}{\csc\theta}+\frac{1}{\sec\theta}\,d\theta$$

$$= \int_0^{\pi/4} \sin\theta+\cos\theta\,d\theta$$

$$= \left[-\cos\theta+\sin\theta\right]_0^{\pi/4}$$

$$= \left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right]-\left[-1+0\right]$$

$$= \boxed{1}.$$

Example Division by rational functions

Evaluate $\displaystyle\int \frac{t^4+t^3+t^2+t+1}{t^2+1}\,dt$

We can do division because $\dfrac{at+b}{t^2+1}$ is integrable for all $a,b \in \mathbb{R}$.

$$
\begin{array}{r|l}
 & t^2 + t \\
\hline
t^2+1 & t^4 + t^3 + t^2 + t + 1 \\
 & -\left(t^4 + 0t^3 + t^2\right) \\
\hline
 & t^3 \qquad + t + 1 \\
 & t^3 \qquad + t \\
\hline
 & \qquad\qquad 1
\end{array}
$$

so $\displaystyle\int \frac{t^4+t^3+t^2+t+1}{t^2+1}\,dt$

$$= \int t^2+t+\frac{1}{t^2+1}\,dt$$

$$= \frac{1}{3}t^3+\frac{1}{2}t^2+\arctan(t) + C$$

<u>Example:</u> Evaluate $\int \frac{t^3-2}{t+1}\,dt$

$$\begin{array}{r|l}
 & t^2-t+1 \\ \hline
t+1 & t^3+0t^2+0t-2 \\
 & t^3+t^2 \\ \hline
 & -t^2+0t-2 \\
 & -t^2-t \\ \hline
 & t-2 \\
 & t+1 \\ \hline
 & -3
\end{array}$$

$$\int \frac{t^3-2}{t+1}\,dt$$

$$= \int t^2-t+1-\frac{3}{t+1}\,dt$$

$$= \frac{1}{3}t^3-\frac{1}{2}t^2+t - 3\ln|t+1| + C$$

Example Complete the Square

Evaluate $\int \frac{dx}{x^2-2x+10}$

$$x^2-2x+10 = (x^2-2x+1)+9$$
$$= (x-1)^2+9$$

$$\int \frac{dx}{x^2-2x+10} = \int \frac{1}{(x-1)^2+9}\,dx$$

$u=(x-1)$
$du=dx$

$$= \int \frac{1}{u^2+9}\,du$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)+C$$

$$= \boxed{\frac{1}{3}\tan^{-1}\left(\frac{(x-1)}{3}\right)+C}$$

Example: Evaluate $\int \frac{d\theta}{(\theta-3)\sqrt{-6\theta+\theta^2}}$

Assume $\theta \geq 6$.

$$-6\theta+\theta^2 = (\theta^2-6\theta+9)-9$$

$$= (\theta-3)^2-9$$

$$\int \frac{d\theta}{(\theta-3)\sqrt{-6\theta+\theta^2}} = \int \frac{d\theta}{(\theta-3)\sqrt{(\theta-3)^2-9}}$$

$u = \theta-3$

$du = d\theta$

$$= \int \frac{du}{u\sqrt{u^2-3^2}}$$

$$= \frac{1}{3}\sec^{-1}\left|\frac{u}{3}\right| + C$$

$$= \boxed{\frac{1}{3}\sec^{-1}\left|\frac{\theta-3}{3}\right| + C}$$