8.2 Integration by Parts

Recall the Fundamental Theorem of Calculus.

Let $f(x) = \frac{d}{dx} F(x)$ Then

$$\boxed{\int f(x)\,dx = \int \frac{d}{dx} F(x)\,dx = F(x) + C.}$$

And recall the product rule says

$$\boxed{\frac{d}{dx}\left(f(x) g(x) \right) = f'(x) g(x) + f(x) g'(x)}$$

combining the two gives

$$\boxed{u(x)v(x) = \int \frac{d}{dx}\left(u(x)v(x)\right)dx = \int u'(x)v(x) + u(x)v'(x)\,dx}$$

This formula is what we call Integration by Parts. However, the more useful form we use in calc II to do integration is the following:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

This comes from the first formula but using $du = u'(x)dx$ and $dv = v'(x)dx$.

Idea: this allows us to switch which term has the derivative, but by switching the derivative we get a negative sign and the uv term.

Example: Compute $\int 2xe^{3x}\,dx$

$u = 2x$ $\quad v = \frac{1}{3}e^{3x}$

$du = 2dx$ $\quad dv = e^{3x}dx$

$$\int 2xe^{3x}\,dx = uv - \int v\,du$$

$$= \frac{2}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,2dx$$

$$= \boxed{\frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + C}$$

Example: Compute $\int x\,3^x\,dx$

$u = x$ $\quad v = \frac{1}{\ln(3)} 3^x$

$du = dx$ $\quad dv = 3^x dx$

$$\int x\,3^x dx = uv - \int v\,du$$

$$= \frac{x3^x}{\ln(3)} - \int \frac{3^x}{\ln(3)} dx$$

$$= \boxed{\frac{x3^x}{\ln(3)} - \frac{3^x}{\ln(3)^2} + C}$$

<u>Example</u>: Compute $\int x^2 e^{4x}\,dx$

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$u = x^2 \qquad v = \frac{1}{4}e^{4x}$

$du = 2x\,dx \qquad dv = e^{4x}\,dx$

$$\int x^2 e^{4x}\,dx = uv - \int v\,du$$

$$= \frac{x^2 e^{4x}}{4} - \int \frac{x}{2} e^{4x}\,dx$$

$u = x \qquad v = \frac{1}{8}e^{4x}$

$du = dx \qquad dv = \frac{1}{2}e^{4x}\,dx$

$$= \frac{x^2 e^{4x}}{4} - \left(uv - \int v\,du\right.$$

$$= \frac{x^2 e^{4x}}{4} - \left(\frac{x e^{4x}}{8} - \int \frac{1}{8} e^{4x}\,dx\right)$$

$$= \boxed{\frac{x^2 e^{4x}}{4} - \frac{x e^{4x}}{8} + \frac{e^{4x}}{32} + C}$$

Example: Compute $\int t^3 \sin t \, dt$

$u = t^3$ $\quad v = -\cos t$

$du = 3t^2 dt$ $\quad dv = \sin t \, dt$

$$\int t^3 \sin t \, dt = -t^3 \cos t + \int 3t^2 \cos t \, dt$$

$u = 3t^2$ $\quad v = \sin t$

$du = 6t \, dt$ $\quad dv = \cos t \, dt$

$$= -t^3 \cos t + 3t^2 \sin t - \int 6t \sin t \, dt$$

$u = 6t$ $\quad v = -\cos t$

$du = 6 \, dt$ $\quad dv = \sin t \, dt$

$$= -t^3 \cos t + 3t^2 \sin t - \left(-6t \cos t + \int 6 \cos t \, dt\right)$$

$$= \boxed{-t^3 \cos t + 3t^2 \sin t + 6t \cos t - 6 \sin t + C}$$

Example: Compute $\int e^x \cos x\,dx$

$u = \cos x \qquad v = e^x$

$du = -\sin x\,dx \qquad dv = e^x dx$

$$\int e^x \cos x\,dx = e^x \cos x + \int \sin x\, e^x dx$$

$u = \sin x \qquad v = e^x$

$du = \cos x\,dx \qquad dv = e^x dx$

$$= e^x \cos x + \sin x\, e^x - \int \cos x\, e^x dx$$

Moving the two $\int \cos x\, e^x dx$ terms to the same side give

$$2\int e^x \cos x\, d = e^x \cos x + e^x \sin x$$

$$\Rightarrow \quad \int e^x \cos x = \boxed{\frac{e^x}{2}(\cos x + \sin x) + C}$$

Example: Compute $\int e^{-x} \sin 4x \, dx$

$u = \sin 4x \qquad v = -e^{-x}$

$du = 4\cos 4x \, dx \qquad dv = e^{-x} dx$

$\int e^{-x} \sin 4x \, dx = -e^{-x} \sin 4x + \int 4e^{-x} \cos 4x \, dx$

$u = \cos 4x \qquad v = -4e^{-x}$

$du = -4 \sin 4x \, dx \qquad dv = 4e^{-x} dx$

$= -e^{-x} \sin 4x - 4e^{-x} \cos 4x - \int 16 \sin 4x e^{-x} dx$

$17 \int e^{-x} \sin 4x = -e^{-x} \sin 4x - 4e^{-x} \cos 4x$

$\int e^{-x} \sin 4x = \boxed{\frac{-e^{-x}}{17} (\sin 4x + 4 \cos 4x)}$

Integration by parts definite integrals

Let u and v be differentiable. Then

$$\boxed{\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx}$$

Example: Suppose $\lim_{|x|\to\infty} u(x) = 0$ and $\lim_{|x|\to\infty} u'(x) = 0$

Then,

$$\int_{-\infty}^{\infty} u(x)\,u''(x) = u(x)u'(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (u'(x))^2\,dx$$

$$= -\int_{-\infty}^{\infty} u'(x)^2\,dx$$

This idea comes up in the study of differential equations.

$$\int_{-\infty}^{\infty} u(x)^2 + u'(x)^2\,dx = \|u\|_{H^1}^2$$

Example: Compute $\int_1^e \ln 2x \, dx$

$u = \ln(2x)$ $\qquad$ $v = x$

$du = \frac{1}{x} dx$ $\qquad$ $dv = dx$

$$\int_1^e \ln(2x)\,dx = x\ln(2x)\Big|_1^e - \int_1^e dx$$

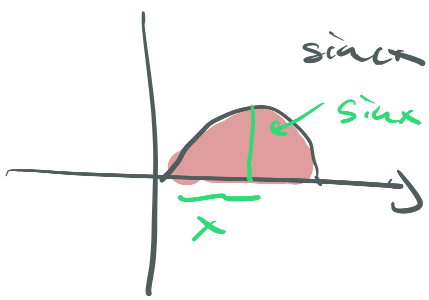
$$= e\ln(2e) - \ln(2) - x\Big]_1^e$$

$$= \boxed{e(\ln(2e) - 1) - (\ln(2) - 1)}$$

$$= e(\ln(2) + \ln(e) - 1) - (\ln(2) - 1)$$

$$= \boxed{e\ln(2) - \ln 2 + 1}$$

Example: Find the volume of the region bounded by $f(x) = \sin x$ and the x-axis on $[0,\pi]$ revolved around the y-axis.



$$\text{Volume} = 2\pi \int_0^{\pi} x \sin x \, dx$$

$u = x \qquad v = -\cos x$

$du = dx \qquad dv = \sin x \, dx$

$$= 2\pi \left[\left[-x\cos x \right]_0^{\pi} - \int_0^{\pi} -\cos x \, dx \right]$$

$$= 2\pi \left[\pi - [\sin x]_0^{\pi} \right]$$

$$= \boxed{2\pi^2}$$