8.3 Trigonometric Integrals

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$

Example:

Compute $\int \sin^5 x \, dx$

$$\int \sin^5 x \, dx = \int (\sin^2 x)^2 \sin x \, dx$$

$\sin^2 x = 1 - \cos^2 x$

$$= \int (1 - u^2 x)^2 \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1 - u^2)^2 \, du$$

$$= -\int 1 - 2u^2 + u^4 \, du$$

$$= -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + C$$

$$= \boxed{-\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C}$$

Example: Compute $\int \cos^3 20x\, dx$

$$\int \cos^3 20x\, dx = \int (\cos^2 20x) \cos 20x\, dx$$

$$= \int (1 - \sin^2 20x) \cos 20x\, dx$$

$u = \sin 20x$

$du = 20 \cos 20x\, dx$

$$= \frac{1}{20} \int (1 - u^2)\, du$$

$$= \frac{1}{20}\left(u - \frac{1}{3}u^3\right) + C$$

$$= \boxed{\frac{1}{20} \sin 20x - \frac{1}{60} \sin^3 20x + C}$$

Half-Angle Formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

<u>Example:</u> Compute $\int \cos^4 x\,dx$

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx$$

$$= \int \left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\right)^2 dx$$

$$= \int \tfrac{1}{4}+\tfrac{2}{4}\cos 2x+\tfrac{1}{4}\cos^2 2x\,dx$$

$$= \int \tfrac{1}{4}+\tfrac{1}{2}\cos 2x+\left(\tfrac{1}{8}+\tfrac{1}{8}\cos 4x\right)dx$$

$$= \int \tfrac{3}{8}+\tfrac{1}{2}\cos 2x+\tfrac{1}{8}\cos 4x\,dx$$

$$= \boxed{\tfrac{3}{8}x+\tfrac{1}{4}\sin 2x+\tfrac{1}{32}\sin 4x + C}$$

Example: Compute $\int \sin^2 3x \, dx$

$$\int \sin^2 3x \, dx = \int \frac{1-\cos 6x}{2} \, dx$$

$$= \frac{1}{2} \int 1 - \cos 6x \, dx$$

$$= \frac{1}{2} \left[x - \frac{1}{6} \sin 6x \right] + C$$

$$= \boxed{\frac{x}{2} - \frac{1}{12} \sin 6x + C}$$

Example: $\int \cos^4 x \sin^2 x \, dx$

$$\int \cos^4 x \sin^2 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)\, dx$$

$$= \frac{1}{8}\int \underbrace{1}_{A} + \underbrace{\cos 2x}_{B} \underbrace{- \cos^2 2x}_{C} \underbrace{- \cos^3 2x}_{D} \, dx$$

$$A = \frac{1}{8}\int 1 \, dx = \frac{x}{8}$$

$$B = \frac{1}{8}\int \cos 2x \, dx = \frac{1}{16}\sin 2x$$

$$C = -\frac{1}{8}\int \cos^2 2x \, dx = -\frac{1}{8}\int \frac{1+\cos 4x}{2}\, dx$$

$$= \frac{-1}{16}\int 1 + \cos 4x \, dx = \frac{-1}{16}\left[x + \frac{1}{4}\sin 4x\right]$$

$$= -\frac{x}{16} - \frac{1}{64}\sin 4x$$

$$D = \frac{-1}{8}\int \cos^3 2x\,dx = \frac{-1}{8}\int (1-\sin^2 2x)\cos 2x\,dx$$

$u = \sin(2x)$

$du = 2\cos(2x)\,dx$

$$= \frac{-1}{16}\int (1-u^2)\,du$$

$$= \frac{-1}{16}\left[u - \frac{1}{3}u^3\right]$$

$$= -\frac{1}{16}\sin 2x + \frac{1}{48}\sin^3 2x$$

$$= \frac{x}{8} + \cancel{\frac{1}{16}\sin 2x} + \left(-\frac{x}{16} - \frac{1}{64}\sin 4x\right)$$

x

$$+\left(\cancel{\frac{-1}{16}\sin 2x} + \frac{1}{48}\sin^3 2x\right) + C$$

$$= \boxed{\frac{x}{16} - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C}$$

Example: Compute $\int \cos^3 x \sin^{-2} x \, dx$

$$\int \cos^3 x \sin^{-2} x \, dx = \int \cos^2 x \sin^{-2} x \cos x \, dx$$

$$= \int (1 - \sin^2 x) \sin^{-2} x \cos x \, dx$$

$u = \sin(x)$

$du = \cos(x)dx$

$$= \int (1 - u^2) u^{-2} \, du$$

$$= \int u^{-2} - 1 \, du$$

$$= -u^{-1} - u + c$$

$$= -\frac{1}{\sin x} - \sin x + c$$

$$= \boxed{-\csc x - \sin x + c}$$

Table 8.2

$\int \sin^m x \cos^n x \, dx$	Strategy
m odd and positive, n real	Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$.
n odd and positive, m real	Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.
m and n both even, nonnegative integers	Use half-angle formulas to transform the integrand into a polynomial in $\cos 2x$, and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Example Compute $\int \sin^2 x \cos^2 x \, dx$

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4}\int 1 - \cos^2 2x \, dx$$

$$= \frac{1}{4}\int 1 - \left(\frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \frac{1}{2} - \frac{\cos 4x}{2} \, dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + C$$

$$= \boxed{\frac{x}{8} - \frac{\sin 4x}{32} + C}$$

Example: Compute $\int \sin^3 x \cos^{3/2} x \, dx$

$$\int \sin^3 x \cos^{3/2} x \, dx = \int \sin^2 x \cos^{3/2} x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^{3/2} x \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1 - u^2) u^{3/2} \, du$$

$$= -\int u^{3/2} - u^{7/2} \, du$$

$$= -\left[\frac{2}{5} u^{5/2} - \frac{2}{9} u^{9/2}\right] + C$$

$$= \boxed{-\frac{2}{5} \cos^{5/2} x + \frac{2}{9} \cos^{9/2} x + C}$$

Reduction Formulas

Assume n is a positive integer.

1. $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

2. $\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

3. $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, \quad n \neq 1$

4. $\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, \quad n \neq 1$

Example: Show formula 3.

$$\int \tan^n x\,dx = \int (\tan^2 x)\tan^{n-2} dx$$

$$= \int (\sec^2 x - 1)\tan^{n-2} x\,dx$$

$$= \int \sec^2 x \tan^{n-2} x\,dx - \int \tan^{n-2} x\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int u^{n-2}\,du - \int \tan^{n-2} x\,dx$$

$$= \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \checkmark$$

Example: compute $\int \tan^5 x \, dx$

$$\int \tan^5 x \, dx = \frac{\tan^4 x}{4} - \int \tan^3 x \, dx$$

$$= \frac{\tan^4 x}{4} - \left[\frac{\tan^2 x}{2} - \int \tan x \, dx \right]$$

14. $\int \tan ax \, dx = \frac{1}{a} \ln |\sec ax| + C$

$$= \boxed{\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + C}$$

Example: Compute $\int \sec^3 x \tan^4 x\,dx$

$$\int \sec^3 x \tan^4 x\,dx = \int \sec^3 x\,(\sec^2 x - 1)^2\,dx$$

$$= \int \sec^3 x\,(\sec^4 x - 2\sec^2 x + 1)\,dx$$

$$= \int \sec^7 x - 2\sec^5 x + \sec^3 x\,dx$$

$$= \frac{\sec^5 x \tan x}{6} + \frac{5}{6}\int \sec^5 x\,dx$$

$$- \int 2\sec^5 x + \sec^3 x\,dx$$

$$= \frac{\sec^5 x \tan x}{6}$$

$$-\frac{7}{6}\int \sec^5 x\,dx - \int \sec^3 x\,dx$$

$$= \frac{\sec^5 x \tan x}{6} - \frac{7}{6}\left[\frac{\sec^3 x \tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx\right]$$

$$-\int \sec^3 x\,dx$$

$$= \frac{\sec^5 x \tan x}{6} - \frac{7\sec^3 x \tan x}{24}$$

$$- \frac{45}{24} \int \sec^3 x \, dx$$

$$= \frac{\sec^5 x \tan x}{6} - \frac{7\sec^3 x \tan x}{24}$$

$$- \frac{45}{24} \left[\frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx \right]$$

$$= \frac{\sec^5 x \tan x}{6} - \frac{7\sec^3 x \tan x}{24}$$

$$- \frac{45 \sec x \tan x}{48} - \frac{45}{48} \ln|\sec x + \tan x|$$

16. $\int \sec ax \, dx = \frac{1}{a} \ln|\sec ax + \tan ax| + C$

$$+ C$$

Example: Compute $\int \sec^6 x \tan^3 x \, dx$

$$\int \sec^6 x \tan^3 x \, dx = \int \tan^3 x \sec^4 x \sec^2 x \, dx$$

$$= \int \tan^3 x (1 + \tan^2 x)^2 \sec^2 x \, dx$$

$$= \int \tan^3 x (1 + 2\tan^2 x + \tan^4 x) \sec^2 x \, dx$$

$$= \int [\tan^3 x + 2\tan^5 x + \tan^7 x] \sec^2 x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u^3 + 2u^5 + u^7 \, du$$

$$= \frac{1}{4}u^4 + \frac{2}{6}u^6 + \frac{1}{8}u^8 + C$$

$$= \boxed{\frac{1}{4}\tan^4 x + \frac{1}{3}\tan^6 x + \frac{1}{8}\tan^8 x + C}$$

Table 8.3

$\int \tan^m x \sec^n x \, dx$	**Strategy**
n even and positive, m real	Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.
m odd and positive, n real	Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.
m even and positive, n odd and positive	Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

Example: Compute $\int 10 \tan^9 x \sec^4 x \, dx$

$$\int 10 \tan^9 x \sec^4 x \, dx = 10 \int \tan^9 x \,(1 + \tan^2 x) \sec^2 x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= 10 \int u^9 (1 + u^2) \, du$$

$$= 10 \int u^9 + u^{11} \, du$$

$$= 10 \left[\frac{1}{10} u^{10} + \frac{1}{12} u^{12} \right] + C$$

$$= \boxed{\tan^{10} x + \frac{5}{6} \tan^{12} x + C}$$

<u>Example</u>: Compute $\int \tan x \sec^3 x \, dx$

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 \sec^2 x \, (\sec x \tan x) \, dx$$

$$= \int (\sec^2 x - 1) \sec^2 x [\sec x \tan x] \, dx$$

$$= \int [\sec^4 x - \sec^2 x][\sec x \tan x] \, dx$$

$u = \sec x$

$du = \sec x \tan x$

$$= \int u^4 - u^2 \, du$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \boxed{\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C}$$