# 8.4 Trigonometric Substitution

Integrals involving $a^2-x^2$.

Idea is to make the substitution

$x = a\sin\theta$ or $x = a\cos\theta$ Then

$$a^2-x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta)$$
$$= a^2\cos^2\theta$$

or

$$a^2-x^2 = a^2 - a^2\cos^2\theta = a^2(1-\cos^2\theta)$$
$$= a^2\sin^2\theta.$$

<u>Example:</u> Compute $\displaystyle\int_0^{3/2} \frac{dx}{(9-x^2)^{3/2}}$

Set $x = 3\sin\theta$ Then $dx = 3\cos\theta\, d\theta$

$\frac{3}{2} = 3\sin\theta \Rightarrow \theta = \pi/6$ $\quad 0 = 3\sin\theta \quad \theta = 0.$

$$\int_0^{3/2} \frac{dx}{(9-x^2)^{3/2}} = \int_0^{\pi/6} \frac{3\cos\theta}{(9-9\sin^2\theta)^{3/2}}\, d\theta$$

$$= \int_0^{\pi/6} \frac{3\cos\theta}{(9\cos^2\theta)^{3/2}}\, d\theta = \int_0^{\pi/6} \frac{3\cos\theta}{27\cos^3\theta}\, d\theta$$

$$= \frac{1}{9}\int_0^{\pi/6} \sec^2\theta\, d\theta = \frac{1}{9} \tan\theta \Big]_0^{\pi/6}$$

$$= \frac{1}{9}\left[\frac{1}{\sqrt{3}} - 0\right] = \boxed{\frac{\sqrt{3}}{27}}$$

<u>Example:</u> Compute $\int \sqrt{64-x^2}\, dx$

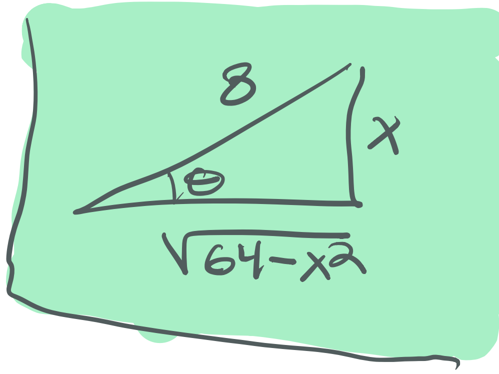
$x = 8\sin\theta \qquad dx = 8\cos\theta$

$\theta = \sin^{-1}(x/8)$

$$\int \sqrt{64-x^2}\, dx = \int \sqrt{64-(8\sin\theta)^2}\; 8\cos\theta\, d\theta$$

$$= \int \sqrt{64(1-\sin^2\theta)}\; 8\cos\theta\, d\theta$$

$$= \int 8\cos\theta \, 8\cos\theta \, d\theta$$




$$= \int 64\cos^2\theta \, d\theta$$

$$= \int 64\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 32 \int 1 + \cos 2\theta \, d\theta$$

$$= 32\theta + 16\sin 2\theta + C$$

$$= 32\arcsin^{-1}\left(\frac{x}{8}\right) + 32\sin\theta\cos\theta + C$$

$$= 32\arcsin^{-1}\left(\frac{x}{8}\right) + 32\,\frac{x}{8}\cdot\frac{\sqrt{64-x^2}}{8} + C$$

$$= \boxed{32\arcsin^{-1}\left(\frac{x}{8}\right) + \frac{x}{2}\sqrt{64-x^2} + C}$$

## Integrals involving $a^2+x^2$.

To reduce this to 1 term we try to set $x = a\tan\theta$ or $x = a\cot\theta$ Then

$$a^2+x^2 = a^2+a^2\tan\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

or $a^2+x^2 = a^2+a^2\cot\theta = a^2(1+\cot^2\theta) = a^2\csc^2\theta$.

Example: Compute $\displaystyle\int \frac{x^2}{(25+x^2)^{3/2}}\,dx$

set $x = 5\tan\theta$ $\qquad dx = 5\sec^2\theta\, d\theta$

$$\int \frac{x^2}{(25+x^2)^{3/2}}\,dx = \int \frac{25\tan^2\theta}{(25+25\tan^2\theta)^{3/2}}\, 5\sec^2\theta\, d\theta$$

$$= \int \frac{25\tan^2\theta}{(25\sec^2\theta)^{3/2}}\, 5\sec^2\theta\, d\theta$$

$$= \int \frac{\tan^2\theta\sec^2\theta}{\sec^3\theta}\, d\theta.$$

$$= \int \tan^2\theta \sec^{-1}\theta \, d\theta$$

$$= \int \tan^2\theta \cos\theta \, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta} \, d\theta$$

$$= \int \frac{(1-\cos^2\theta)}{\cos\theta} \, d\theta$$

$$= \int \sec\theta - \cos\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$\tan\theta = \frac{x}{5}$$

$$\sin\theta = \frac{x}{\sqrt{25+x^2}}$$

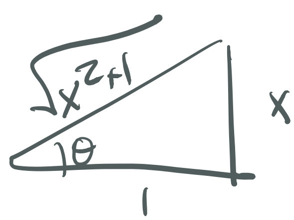
$$\cos\theta = \frac{5}{\sqrt{25+x^2}}$$

$$= \boxed{\ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| - \frac{x}{\sqrt{25+x^2}} + C}$$

**Example:** Compute $\int \frac{dx}{(1+x^2)^{3/2}}$

$x = \tan\theta \qquad dx = \sec^2\theta \, d\theta$

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\sec^2\theta}{(1+\tan^2\theta)^{3/2}} \, d\theta$$




$$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}} \, d\theta$$

$$= \int \frac{1}{\sec\theta} \, d\theta$$

$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + C$$

$$= \boxed{\frac{x}{\sqrt{x^2+1}} + C}$$

## Integrals of the form $x^2-a^2$

To reduce this to 1 term we try

$x = a\sec\theta$ or $x = a\csc\theta$ then

$$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2(1-\sec^2\theta) = a^2\tan^2\theta$$

or

$$x^2-a^2 = a^2\csc^2\theta - a^2 = a^2(1-\csc^2\theta) = a^2\cot^2\theta.$$

**Example:** Compute $\int \frac{dx}{x^2-6x-16}$ For $x>8$

$$x^2-6x-16 = (x^2-6x+9)-25$$
$$= (x-3)^2-25$$

Note:
$x^2-6x-16$
$= (x-8)(x+2)$
so we have to restrict to $x>8$ to avoid dividing by $0$

$$\int \frac{dx}{x^2-6x-16} = \int \frac{1}{(x-3)^2-25}\,dx$$

set $(x-3) = 5\sec\theta$ $\quad dx = 5\sec\theta\tan\theta\,d\theta$

$$= \int \frac{1}{25\sec^2\theta-25}\,5\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{5\sec\theta\tan\theta}{25\tan^2\theta}\,d\theta$$

$$= \frac{1}{5}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{5}\int \frac{1}{\cos\theta\,\frac{\sin\theta}{\cos\theta}}\,d\theta$$

$$= \frac{1}{5}\int \frac{1}{\sin\theta}\,d\theta = \frac{1}{5}\int \csc\theta\,d\theta$$

$$= \frac{-1}{5}\ln\left|\csc\theta + \cot\theta\right|$$

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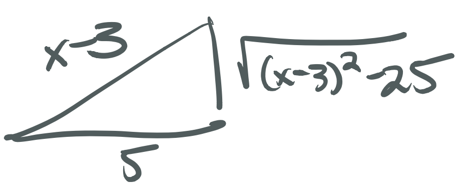
$$(x-3) = 5\sec\theta \Rightarrow \frac{x-3}{5} = \frac{1}{\cos\theta}$$

$$\cos\theta = \frac{5}{x-3} \qquad \sin\theta = \frac{\sqrt{(x-3)^2-25}}{x-3}$$

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$$= \boxed{\frac{-1}{5}\ln\left|\frac{x-3}{\sqrt{(x-3)^2-25}} + \frac{5}{\sqrt{(x-3)^2-25}}\right| + C}$$

**Remark:** This solution only makes sense when $x > 8$ so we have $(x-3)^2 - 25 > 0$.

**Example:** Compute $\int \frac{dx}{x^2\sqrt{9x^2-1}}$, $x > 1/3$

$x = \frac{1}{3}\sec\theta \qquad d\theta = \frac{1}{3}\sec\theta\tan\theta$

$$\int \frac{dx}{x^2\sqrt{9x^2-1}} = \int \frac{1}{(\frac{1}{3}\sec\theta)^2\sqrt{9(\frac{1}{3}\sec\theta)^2-1}} \frac{1}{3}\sec\theta\tan\theta\, d\theta$$

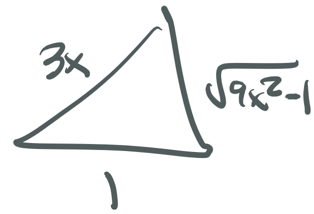
$$= \int \frac{9}{\sec^2\theta\sqrt{\sec^2\theta-1}} \frac{1}{3}\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{3\cancel{\sec\theta}\,\cancel{\tan\theta}}{\cancel{\sec^2\theta}\,\cancel{\tan\theta}}\, d\theta$$

$$= \int \frac{3}{\sec\theta}\, d\theta = \int 3\cos\theta\, d\theta$$

$$= 3\sin\theta + C$$

$\sec\theta = \frac{3x}{1}$ $\qquad$ $\cos\theta = \frac{1}{3x}$




$\sin\theta = \frac{\sqrt{9x^2-1}}{3x}$

$$= \boxed{\frac{\sqrt{9x^2-1}}{3x} + C}$$

## Integrals Involving $a^2 + x^2$ or $x^2 - a^2$

The additional trigonometric substitutions involving tangent and secant use a procedure similar to that used for the sine substitution. Figure 8.5 and Table 8.4 summarize the three basic trigonometric substitutions for real numbers $a > 0$.

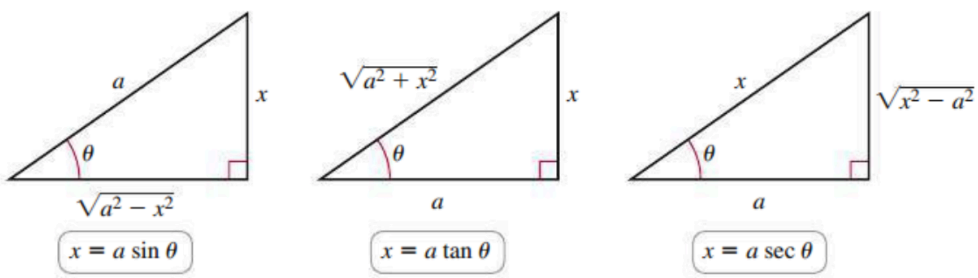



**Figure 8.5**

**Table 8.4**

| The Integral Contains . . . | Corresponding Substitution | Useful Identity |
|---|---|---|
| $a^2 - x^2$ | $x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for $\lvert x\rvert \le a$ | $a^2 - a^2\sin^2\theta = a^2\cos^2\theta$ |
| $a^2 + x^2$ | $x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ | $a^2 + a^2\tan^2\theta = a^2\sec^2\theta$ |
| $x^2 - a^2$ | $x = a\sec\theta, \begin{cases} 0 \le \theta < \frac{\pi}{2}, \text{ for } x \ge a \\ \frac{\pi}{2} < \theta \le \pi, \text{ for } x \le -a \end{cases}$ | $a^2\sec^2\theta - a^2 = a^2\tan^2\theta$ |

<u>Remark</u>: With these integral substitutions we have to be careful with the domain. $x = a\sin\theta$ only gives values for $|x| \le a$. This is not a problem when we have something like $\sqrt{a^2-x^2}$ since this is defined for $|x| \le a$.

However, in a case like $\frac{1}{a^2-x^2}$ this is defined for $|x| > a$. So we can rewrite this as $\frac{-1}{x^2-a^2}$ and use $x = a\sec\theta$ since this is defined for $|x| \ge a$.