Partial Fractions

February 5, 2018

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Gives

$$\int \frac{x^3}{x^2 - 2x - 3} dx = \int \left(x + 2 + \frac{7x + 6}{x^2 - 2x - 3} \right) dx.$$

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Example:

$$\int \frac{x}{x^2 - 2x - 3} dx = \int \frac{x}{(x-3)(x+1)} dx = \int \left(\frac{A}{x-3} + \frac{B}{x+1} \right) dx.$$

we need $\dfrac{A}{x-3} + \dfrac{B}{x+1} = \dfrac{x}{(x-3)(x+1)}$

so, $\dfrac{(x+1)A + (x-3)B}{(x-3)(x+1)} = \dfrac{x}{(x-3)(x+1)}$

so, $Ax + Bx = x$ and

$A - 3B = 0$

$\Rightarrow$ $A = 3B$

$4Bx = x \Rightarrow B = 1/4$

$A = 3/4$

$$\int \frac{x}{x^2-2x-3}\,dx = \int \frac{3/4}{x-3} + \frac{1/4}{x+1}\,dx$$

$$= \boxed{\frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + C}$$

Example: Compute $\int \frac{x^2+12x-4}{x^3-4x}\,dx$

$x^3 - 4x = x(x+2)(x-2)$

$$\frac{x^2+12x-4}{x^3-4x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$A(x+2)(x-2) + B(x-2)x + C(x+2)x$$

$$= A(x^2-4) + B(x^2-2x) + C(x^2+2x)$$

$$= x^2 + 12x - 4$$

so,

$A + B + C = 1$ $\quad B + C = 0 \Rightarrow B = -C$

$-2B + 2C = 12$ $\quad 4C = 12 \quad C = 3$

$-4A = -4$ $\quad A = 1 \quad B = -3$

$$\int \frac{x^2+12x-4}{x^3-4x}\,dx = \int \frac{1}{x} - \frac{3}{x+2} + \frac{3}{x-2}\,dx$$

$$= \boxed{\ln|x| - 3\ln|x+2| + 3\ln|x-2| + C.}$$

Repeated linear factors

$Q(x) = (x - a)^n$. Then use

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_n}{(x-a)^n}.$$

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Example:

$$\int \frac{2x+5}{x^2+4x+4} dx = \int \frac{2x+5}{(x+2)^2} dx = \int \left(\frac{A}{x+2} + \frac{B}{(x+2)^2} \right) dx.$$

$$\frac{2x+5}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$2x+5 = (x+2)A + B$$

$$A = 2$$

$$2A + B = 5 \quad \Rightarrow B = 1$$

$$\int \frac{2x+5}{x^2+4x+4}\,dx = \int \frac{2}{x+2} + \frac{1}{(x+2)^2}\,dx$$

$$= \boxed{2\ln|x+2| - \frac{1}{(x+2)} + C}$$

Example: Compute $\displaystyle\int \frac{16x^2}{(x-6)(x+2)^2}\,dx$

$$\frac{16x^2}{(x-6)(x+2)^2} = \frac{A}{x-6} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$A(x+2)^2 + B(x-6)(x+2) + C(x-6) = 16x^2$$

$$A(x^2+4x+4) + B(x^2-4x-12) + C(x-6) = 16x^2$$

$A+B=16$

$4A-4B+C=0$

$4A-12B-6C=0$ } $\Rightarrow$

$A+B=16$

$28A-36B=0$

$7A=9B$

$A=\frac{9B}{7}$

$\frac{16B}{7}=16$ $\quad B=7$ $\quad A=9.$

$36-28+C=0$ $\quad C=-8$

$A=9$	$B=7$	$C=-8$

$$\int \frac{16x^2}{(x-6)(x+2)^2}\,dx = \int \frac{9}{x-6} + \frac{7}{(x+2)} - \frac{8}{(x+2)^2}\,dx$$

$$= \boxed{9\ln|x-6| + 7\ln|x+2| + \frac{8}{x+2} + C}$$

Irreducible quadratic

If $Q = ax^2 + bx + c$ has no real roots ($b^2 - 4ac < 0$), try

$$\frac{Ax + B}{ax^2 + bx + c},$$

and complete the square. Then use u substitution and arctan.

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Example:

$$\int \frac{x - 3}{x^2 + 2x + 5} dx =$$

$$\frac{x-3}{x^2+2x+5} = \frac{x+1}{x^2+2x+5} + \frac{-4}{x^2+2x+5}$$

Why? Note that if $u = x^2+2x+5$

$du = 2x+2\,dx$

so $x+1\,dx = \frac{1}{2}du$

and we can do a u-substitution

$$\int \frac{x-3}{x^2+2x+5} = \int \frac{x+1}{x^2+2x+5}dx - \int \frac{4}{x^2+2x+5}dx$$

$$= \int \frac{1}{u}\,\frac{1}{2}du - \int \frac{4}{x^2+2x+5}$$

$$= \frac{1}{2}\ln|u| - \int \frac{4}{x^2+2x+5}dx$$

$$= \frac{1}{2}\ln|x^2+2x+5| - \int \frac{4}{x^2+2x+5}dx$$

Now with $\frac{4}{x^2+2x+5}$ we complete the square

$$\frac{4}{x^2+2x+5} = \frac{4}{(x+1)^2+4}$$

$u = (x+1)$

$du = dx$

$$\int \frac{4}{(x+1)^2+4}\,dx = \int \frac{4}{u^2+4}\,dx$$

$$= \frac{4}{2}\arctan\left(\frac{u}{2}\right)$$

$$= 2\arctan\left(\frac{x+1}{2}\right)$$

Combining everything

$$\int \frac{x-3}{x^2+2x+5}\,dx = \frac{1}{2}\ln|x^2+2x+5| - 2\arctan\left(\frac{x+1}{2}\right) + C$$

Strategy for partial fractions

Step 1: Is $\deg(P) \geq \deg(Q)$? If so, long division.
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Example:

$$\int \frac{3x-10}{x^2(x^2+4x+6)}\,dx.$$

Use:

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4x+6}.$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4x+6} = \frac{3x-10}{x^2+4x+6}$$

$$Ax(x^2+4x+6) + B(x^2+4x+6) + x^2(Cx+D)$$
$$= 3x - 10$$

Trick: substitute in values for x.

If $x=0$ then $6B = -10$

$B = -5/3$

$A + C = 0$ — x^3 term

$4A + B + D = 0$ — x^2 term

$6A + 4B = 3$ — x term

$6B = -10$ — Constant term

$B = -5/3$

$6A = 3 - 4B = 3 + \frac{20}{3} = \frac{29}{3}$ $A = \frac{29}{18}$

$C = \frac{-29}{18}$ $D = -\frac{58}{9} + \frac{15}{9} = -\frac{43}{9} = -\frac{86}{18}$

$A = 13/16 \quad B = -5/8 \quad C = -13/16 \quad D = -3/4$

$$\int \frac{3x-10}{x^2(x^2+4x+6)}dx. = \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4x+6}\,dx$$

$$= \int \frac{29/18}{x} + \frac{-5/3}{x^2} + \frac{-\frac{29}{18}x - \frac{86}{18}}{x^2+4x+6}\,dx$$

$$= \frac{1}{18}\int \frac{29}{x} - \frac{30}{x^2} - \frac{29x+86}{x^2+4x+6}\,dx$$

$$= \frac{1}{18}\left[29\ln|x| + \frac{30}{x}\right] - \frac{1}{18}\int \frac{29x+86}{x^2+4x+6}\,dx$$

$$\int \frac{29x+58}{x^2+4x+6}\,dx = \int \frac{29x+58}{x^2+4x+6} + \frac{28}{x^2+4x+6}\,dx$$

$$= \ln|x^2+4x+6| + \int \frac{28}{(x+2)^2+2}\,dx$$

$$= \frac{29}{2}\ln|x^2+4x+6) + \frac{28}{\sqrt{2}}\arctan\left(\frac{x+2}{\sqrt{2}}\right)$$

So,

$$\int \frac{3x-10}{x^2(x^2+4x+6)}dx. = \frac{29}{18}\ln|x| + \frac{5}{3x}$$

$$- \frac{29}{36}\ln|x^2+4x+6| - \frac{28}{18\sqrt{2}}\arctan\left(\frac{x+2}{\sqrt{2}}\right) + C$$

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$$\int \frac{x^3 + 2}{x(x-1)^2(x^2 + 4x + 6)} dx$$

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Example:

$$\int \frac{x^3 + 2}{x(x-1)^2(x^2 + 4x + 6)} dx$$

Use:

$$\frac{A_1}{x} + \frac{A_2}{x - 1} + \frac{A_3}{(x-1)^2} + \frac{A_4 x + A_5}{x^2 + 4x + 6}.$$

Example: $\int \frac{8(x^2+4)}{x(x^2+8)}\,dx$

$$\frac{(x^2+4)}{x(x^2+8)} = \frac{A}{x} + \frac{Bx+C}{x^2+8}$$

$$A(x^2+8) + Bx^2 + Cx = x^2 + 4$$

$A = 1/2$ $C = 0$ $B = 1/2$

$$\int \frac{8(x^2+4)}{x(x^2+8)} = 8\int \frac{1/2}{x} + \frac{1/2\, x}{x^2+8}\,dx$$

$$= 4\ln|x| + 4\int \frac{x}{x^2+8}\,dx$$

$$= \boxed{4\ln|x| + 2\ln|x^2+8| + C}$$