Integration Strategy

February 7, 2018

Strategy steps

1. Simplify if possible
2. Look for "obvious" *u*-subs
3. Classify to decide what technique(s) to try
4. Try again!

Simplify if possible

Examples:

$$\int \frac{(x^2-4)^{1/2}}{\sqrt{x-2}}dx = \int \frac{(x-2)^{1/2}(x+2)^{1/2}}{(x-2)^{1/2}}dx = \int (x+2)^{1/2}dx.$$

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$$\int \ln(\sqrt{x^2-4})dx = \int \ln((x^2-4)^{1/2})dx = \int \frac{1}{2}\ln((x-2)(x+2))dx$$

$$= \int \frac{1}{2}(\ln(x-2)+\ln(x+2))dx.$$

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$$\int \sin(2x)\cos x dx = \int (2\sin x\cos x)\cos x dx = 2\int \cos^2 x \sin x dx.$$

“Obvious” substitutions

Here we mean *u*-sub. Examples

$$\int xe^{x^2}dx, \;\; u = x^2, du = 2xdx$$

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$$\int \frac{\cos(\ln(x^2))}{x}dx = \int \frac{\cos(2\ln(x))}{x}dx, \;\; u = 2\ln(x), du = \frac{2}{x}dx$$

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Use it to change the problem ($u = x - 3$):

$$\int \frac{x}{(x-3)^2+4}dx = \int \frac{u+3}{u^2+4}du = \int \frac{u}{u^2+4}du + 3\int \frac{1}{u^2+4}dx$$

Classify integral type

u-substitution, integration by parts

Trigonometric Integrals, Trig substitutions

Partial Fractions.

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$$\int \frac{x}{1+x^2}dx$$

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$\rightsquigarrow$ u-sub

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IBP twice to get integral in terms of itself.

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Trig integrals. For sin and cos, if $n = 2k + 1$ is odd, write $\sin^{2k+1} x \cos^m x = \sin x (\sin^2 x)^k \cos x = \sin x (1 - \cos^2 x)^k \cos^m x$, and u-sub $u = \cos x$, $du = -\sin x dx$. Similar if m odd.
For tan and sec, if $m = 2k$ is even, reserve $\sec^2 x$, write $\sec^2 x = 1 + \tan^2 x$ and u-sub $\tan x$. If $n = 2k + 1$ is odd, keep $\sec x \tan x$, and sub $\tan^2 x = \sec^2 x - 1$, u-sub $u = \sec x$.

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$$\int \cos^3 x \sin^{10} x dx = \int \cos^2 x \sin^{10} x \cos x dx$$

$$= \int (1 - \sin^2 x) \sin^{10} x \cos x dx = \int (1 - u^2) u^{10} du$$

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$(a^2 + x^2)^{1/2}$ use $x = a\tan\theta$, so
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Repeated linear factor:

$$\frac{P(x)}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2},$$

Irreducible quadratic

Irreducible quadratic: $ax^2 + bx + c$ with $b^2 - 4ac < 0$.
Complete the square. For integral, use u-sub, then arctan.

$$\frac{2x-3}{x^2-4x+6} = \frac{2x-3}{x^2-4x+4-4+6} = \frac{2x-3}{(x-2)^2+2}$$

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$u = x - 2$, $du = dx \rightsquigarrow$

$$\int \frac{2u+1}{u^2+2} du = \int \frac{2u}{u^2+2} du + \int \frac{1}{u^2+2} du$$

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first one is a substitution, second one uses arctan.

Which technique should you use on the following integrals?

A) $\int \cos^2 10x \, dx$

B) $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$

C) $\int \frac{x^{-2}+x^{-3}}{x^{-1}+16x^{-3}} dx$

D) $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}} dx$

E) $\int \frac{dx}{x^4-1}$

F) $\int \frac{3x^2+3x+1}{x^3+x} dx$

G) $\int \sin\sqrt{x} \, dx$

H) $\int e^x \cot^3 e^x \, dx$

A) Trig integral half-angle formula

B) $u = e^x$ then trig substitution

C) Multiply by x^3 then u-substitution

D) u-substitution

E) Partial Fractions

F) Partial Fractions

u=sqrt{x} then integration by parts

G) Integration by parts

$u = \sin(\sqrt{x})$ $dv = dx$

H) u-sub then trig integral.

Example

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Now split into two integrals and $u-$sub. First integral (second one similar):

$$\frac{1}{2} \int \ln(x+2)dx = \frac{1}{2} \int \ln(u)du$$

Example cont'd

Now it is IBP: We already used u, so let's use w so $\int w dv = wv - \int v dw$. $w = \ln(u)$, $dv = du$, $dw = (1/u)du$, $v = u$:

$$= wv - \int v dw = \frac{1}{2} u \ln(u) - \frac{1}{2} \int u(1/u) du = \frac{1}{2}(u \ln u - u) + C$$

Sub back for x:

$$= \frac{1}{2}((x+2)\ln(x+2) - (x+2)) + C.$$

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Do it again!

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First integral $w = u^2 + 2,\ dw = 2udu$
$\rightsquigarrow \int (1/w)dw = \ln(|w|) + C$. Sub back
$w = u^2 + 2 = (x-2)^2 + 2.$

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$w = u^2+2 = (x-2)^2+2$. Second one: $\frac{1}{u^2+2} = \frac{1}{2}\left(\frac{1}{1+u^2/2}\right)$, so
sub $w = u/\sqrt{2}$, $dw = (1/\sqrt{2})du \rightsquigarrow$

$$\frac{1}{2}\sqrt{2}\int \frac{1}{1+w^2}dw = \frac{1}{\sqrt{2}}\arctan(w) + C,$$

and sub back $w = u/\sqrt{2} = (x-2)/\sqrt{2}$.

Example: $\int \frac{1}{e^{2x}+1}\,dx$

$$\int \frac{1}{e^{2x}-1}\,dx = \int \frac{e^{2x}-e^{2x}+1}{e^{2x}-1}\,dx$$

$$= \int \frac{-(e^{2x}-1)}{e^{2x}-1} + \frac{e^{2x}}{e^{2x}-1}\,dx$$

$$= \int -1 + \frac{e^{2x}}{e^{2x}-1}\,dx$$

$$= \boxed{-x + \frac{1}{2}\ln\left|e^{2x}-1\right| + C}$$