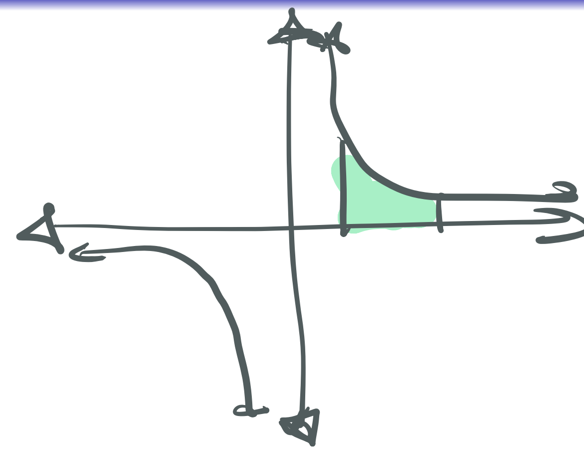
Improper Integrals

February 12, 2018

Motivation

What is

$$\int_1^2 \frac{1}{x} dx?$$



$$\int_1^2 \frac{1}{x} dx = \ln|x| \Big]_1^2 = \boxed{\ln 2}$$

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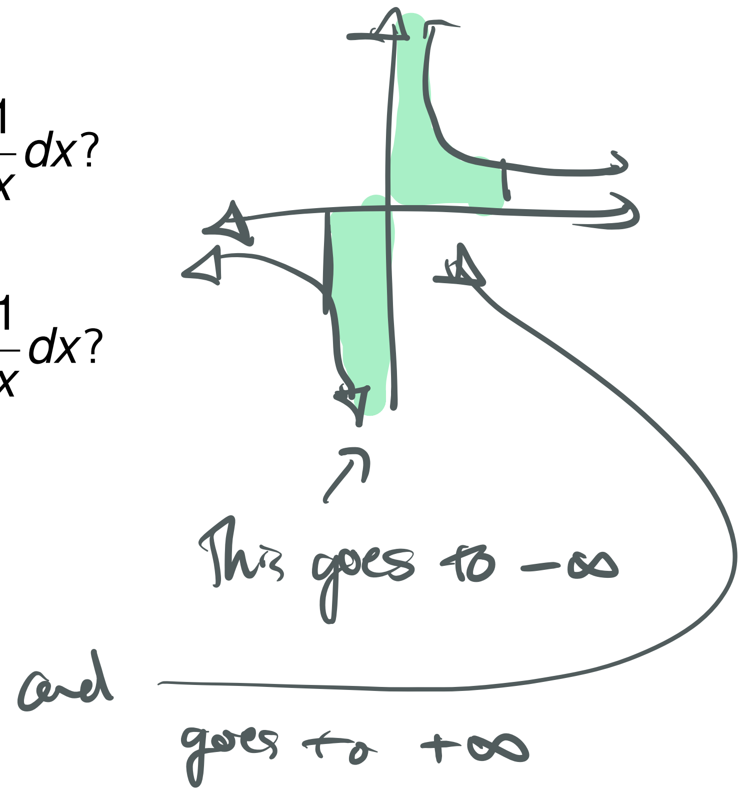

Motivation

What is

$$\int_1^2 \frac{1}{x} dx?$$

What is

$$\int_{-1}^2 \frac{1}{x} dx?$$

Second one is meaningless since $1/x$ is not continuous at $x = 0$.

Motivation cont'd

What is

$$\int_0^\infty \frac{1}{1+x^2}dx?$$

Motivation cont'd

What is

$$\int_0^\infty \frac{1}{1+x^2}\,dx?$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

What does it mean to evaluate at ∞?

The idea is "$\arctan(\infty)$" $= \lim_{x\to\infty} \arctan(x)$

So $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{x\to\infty}\left[\arctan(x) - \arctan(0)\right]$

$= \boxed{\pi/2}$

Motivation cont'd

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$$\int_0^\infty \frac{1}{1+x^2}\,dx?$$

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$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx?$$

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What does it mean to evaluate at ∞?

What is

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx?$$

$$\int x^{-1/2}\,dx = 2x^{1/2} + C.$$

What does it mean to evaluate at ∞? Second one is not as nice...

Example

$$\int_0^\infty \frac{1}{1+x^2}dx :$$

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Then $\int_1^\infty \frac{1}{1+x^2}dx$ converges, and converges to $\pi/2$.

Type 1 example

$$\int_{1}^{\infty} x^{-1/2} dx$$

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Diverges.

Exercise

Determine whether the following integral converges or diverges, and compute the integral if it converges:

$$\int_1^{\infty} x^{-2} dx$$

$$\int_1^{\infty} x^{-2} dx = \lim_{k \to \infty} \int_1^{k} x^{-2} dx = \lim_{k \to \infty} \left[-x^{-1} \right]_1^{k}$$

$$= \lim_{k \to \infty} \left[-\frac{1}{k} - (-1) \right] = \lim_{k \to \infty} \left[1 - \frac{1}{k} \right] = \boxed{1}$$

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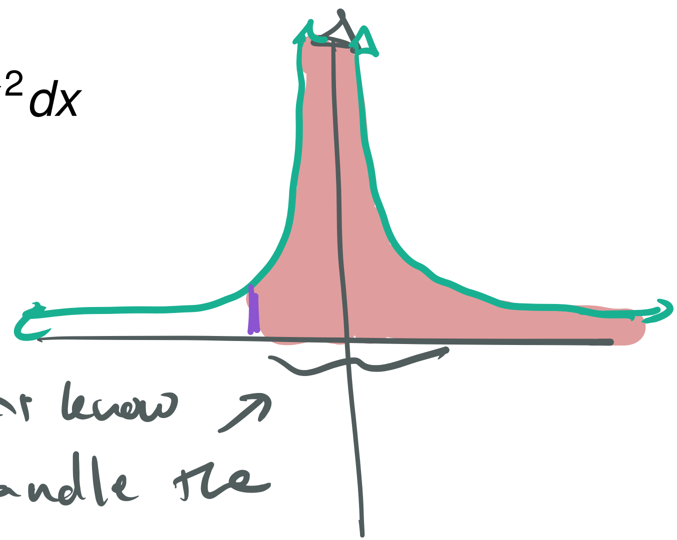
What about $\int_{-1}^{\infty} x^{-2}dx$?



we don't know how to handle the issue at $x = 0$ yet.

Exercise

Determine whether the following integral converges or diverges, and compute the integral if it converges:

$$\int_{1}^{\infty} x^{-2} dx$$

What about $\int_{-1}^{\infty} x^{-2} dx$? $\int_{-1}^{0} x^{-2} dx$ does not make sense (yet).

p-test

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges for $p > 1$ and diverges for $p \leq 1$.

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converges for $p > 1$ and diverges for $p \leq 1$.

$$\int_1^t x^{-p} dx = \begin{cases} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}, & p \neq 1, \\ \ln|t| - \ln 1, & p = 1. \end{cases}$$

This converges if and only if $p > 1$ so that $1 - p$ is a negative power.

This will be important for series in the upcoming chapters.

Type 2 example

$$\int_0^1 \frac{1}{x^p} dx$$

converges for $p < 1$ and diverges for $p \geq 1$.

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converges for $p < 1$ and diverges for $p \geq 1$.

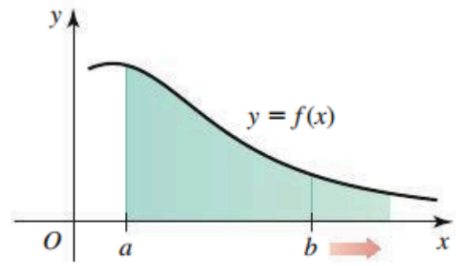
$$\int_t^1 x^{-p} dx = \begin{cases} \frac{1}{1-p} - \frac{t^{1-p}}{1-p}, & p \neq 1, \\ \ln 1 - \ln|t|, & p = 1 \end{cases}$$

Converges if $1 - p$ is a positive power.

DEFINITION Improper Integrals over Infinite Intervals

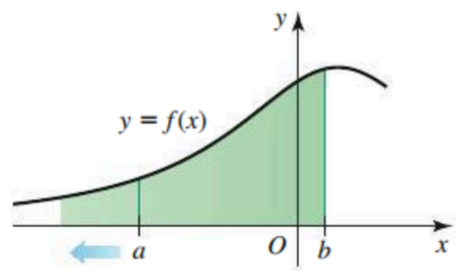
1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$



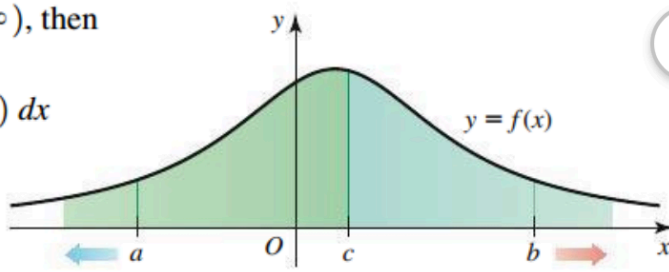
2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$



3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a \to -\infty} \int_a^c f(x)\,dx + \lim_{b \to \infty} \int_c^b f(x)\,dx,$$



where c is any real number. It can be shown that the choice of c does not affect the value or convergence of the original integral.

If the limits in cases 1–3 exist, then the improper integrals **converge**; otherwise, they **diverge**.

<u>Example:</u> Compute $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

We already showed $\int_0^\infty \frac{1}{1+x^2}\,dx$ exists.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to\infty} \int_0^a \frac{1}{1+x^2}\,dx$$

$$+ \lim_{b\to-\infty} \int_b^0 \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to\infty} \Big[\arctan(x)\Big]_0^a + \lim_{b\to-\infty} \Big[\arctan(x)\Big]_b^0$$

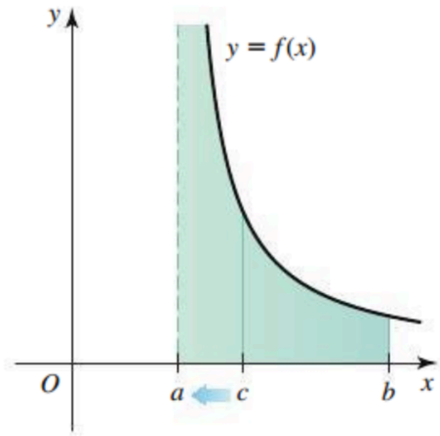
$$= \lim_{a\to\infty} \arctan(a) + \lim_{b\to-\infty} -\arctan(b)$$

$$= \frac{\pi}{2} + -\left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

DEFINITION Improper Integrals with an Unbounded Integrand

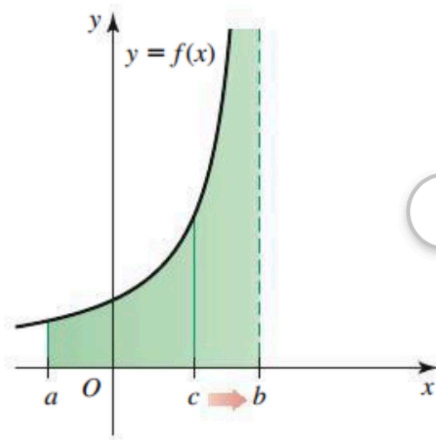
1. Suppose f is continuous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$



2. Suppose f is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$. Then

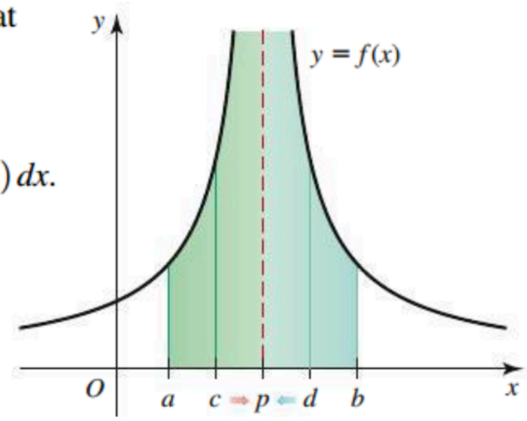
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$



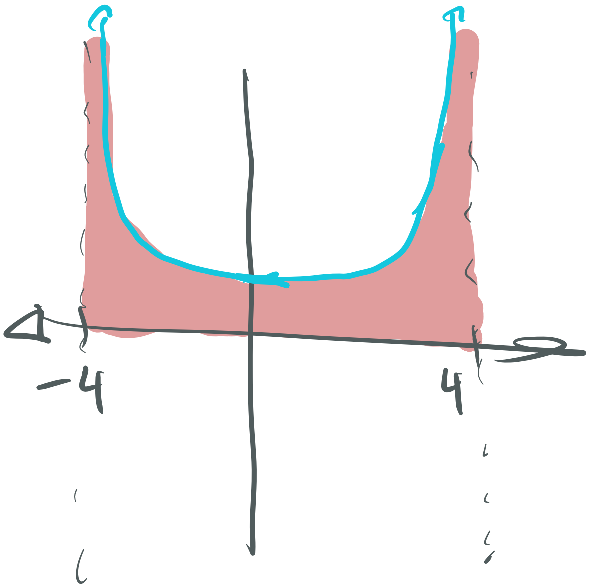
3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \lim_{c\to p^-} \int_a^c f(x)\,dx + \lim_{d\to p^+} \int_d^b f(x)\,dx.$$

If the limits in cases 1–3 exist, then the improper integrals **converge**; otherwise, they **diverge**.



Example: Compute $\int_{-4}^{4} \frac{1}{\sqrt{16-x^2}}\,dx$



$$\int_{-4}^{4} \frac{1}{\sqrt{16-x^2}}\,dx$$

$$= \lim_{a \to -4^+} \int_{a}^{0} \frac{1}{\sqrt{16-x^2}}\,dx$$

$$+ \lim_{b \to 4^-} \int_{0}^{b} \frac{1}{\sqrt{16-x^2}}\,dx$$

$$= \lim_{a \to -4^+} \left[\sin^{-1}(0) - \sin^{-1}\left(\frac{a}{4}\right) \right]$$

$$+ \lim_{b \to 4^-} \left[\sin^{-1}\left(\frac{b}{4}\right) - \sin^{-1}(0) \right]$$

$$= -\left(\frac{-\pi}{2}\right) + \frac{\pi}{2} = \boxed{\pi}$$

THEOREM 8.2 Comparison Test for Improper Integrals
Suppose the functions f and g are continuous on the interval $[a, \infty)$, with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
2. If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example: Show that $\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$, $a>1$ converges.

From above we showed that

$$\int_0^\infty \frac{1}{x^2+1} < \infty.$$ Now if $a>1$

then $\frac{1}{x^2+a^2} < \frac{1}{x^2+1}$ so by the comparison test $\int_0^\infty \frac{1}{x^2+a^2}\,dx \leq \int_0^\infty \frac{1}{x^2+1} < \infty.$

Similarly $\int_{-\infty}^0 \frac{1}{x^2+1} < \infty.$ and $\frac{1}{x^2+a^2} < \frac{1}{x^2+1}$

so by the comparison test

$$\int_{-\infty}^0 \frac{1}{x^2+a^2} \leq \int_{-\infty}^0 \frac{1}{x^2+1} < \infty.$$ So

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx$$ converges.

<u>Remark</u>: This converges to $\frac{\pi}{a}$.

<u>Example</u>: Does the following integral converge or diverge?

(a) $\int_1^{\infty} \frac{1}{x^3+1}\,dx \le \int_1^{\infty} \frac{1}{x^3}\,dx$ converge

(b) $\int_0^{\infty} \frac{1}{e^x+x+1}\,dx \le \int_0^{\infty} \frac{1}{e^x}$ converge

(c) $\int_3^{\infty} \frac{1}{\ln x}\,dx \ge \int_3^{\infty} \frac{1}{x}$ diverge

(d) $\int_0^{1} \frac{\sin x+1}{x^5}\,dx \ge \int_0^{1} \frac{1}{x^5}$ diverge