Power series

July 19, 2021

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is called a power series **centered at** a. (Since $(x-a)$ shows up in powers).
Example:

$$\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}$$

However $\sum_{n=0}^{\infty} \frac{\sin(n)}{n^2} x^n$ is.

is *not* a power series.

Examples

Our favorite series is the geometric series: For $|x| < 1$, *power series*

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$$f(x) = \frac{1}{1-x}.$$

We say $\sum_0^\infty x^n$ *represents* $f(x)$ on the interval $(-1, 1)$. $f(x)$ makes sense for $x \neq 1$, but power series only makes sense for x in $(-1, 1)$.

Example

Find a function represented by the series

$$\sum_{n=0}^{\infty}(3x-2)^n$$

$$\sum_{n=0}^{\infty}(3x-2)^n = \frac{1}{1-(3x-2)} = \frac{1}{3-3x}$$

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$$\sum_{n=0}^{\infty}(3x-2)^n = \frac{1}{1-(3x-2)} = \frac{1}{3-3x}$$

Converges when $|r| = |3x-2| < 1$, or $|x - 2/3| < 1/3$.

Find a function represented by

$$\sum_{n=1}^{\infty}(-1)^n(2x+1)^n$$

For which values of x does the series converge?

we need $\lim_{n\to\infty} |2x+1|^n = 0$ so $0 \leq |2x+1| < 1$

so, $-\frac{1}{2} < x < 0$.

$$\sum_{n=1}^{\infty}(-2x-1)^n = (-2x-1)\sum_{n=0}^{\infty}(-2x-1)^n$$

$$= (-2x-1)\left(\frac{1}{1-(-2x-1)}\right) = \frac{-2x-1}{2x+2} = -\left(\frac{2x+1}{2x+2}\right)$$

Exercise

Find a function represented by

$$\sum_{n=1}^{\infty} (-1)^n (2x+1)^n$$

For which values of x does the series converge?
Geometric:

$$r = -(2x+1),$$

so converges for $|x+1/2| < 1/2$. Converges to:

$$f(x) = \sum_{n=1}^{\infty} (-(2x+1))^n = (-(2x+1)) \sum_{n=0}^{\infty} (-(2x+1))^n$$
$$= \frac{-(2x+1)}{1-(-(2x+1))} = -\frac{2x+1}{2+2x}$$

on $(-1/2-1/2, -1/2+1/2) = (-1, 0)$.

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Ratio test: $a_n = \frac{(3x)^n}{n!}$ for $n \geq 12$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(3x)^{n+1}}{(n+1)!}}{\frac{(3x)^n}{n!}}\right| = \left|\frac{(3x)^{n+1} n!}{(n+1)!(3x)^n}\right|$$
$$= \frac{3x}{n+1} \to 0$$

for all x. Series converges for every x.

Exercise

For which values of x does the following series converge?

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Ratio test: $a_n = \frac{(4x-9)^{2n}}{(n+1)!}$

$$\left| \frac{\frac{(4x-9)^{2(n+1)}}{((n+1)+1)!}}{\frac{(4x-9)^{2n}}{(n+1)!}} \right| = \frac{|(4x-9)^2|}{n+2} \to 0,$$

no matter what x is.

Radius of convergence

Definition/Theorem: Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series centered at a. Exactly one is true:
1) series diverges (except when $x = a$)
2) series converges for all x
3) there is a number $R > 0$ such that series converges for $|x - a| < R$ and diverges for $|x - a| > R$.

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Examples:

1) $\sum_{n=1}^{\infty} n!(x-2)^n$ converges only at $x = 2$
2) $\sum_{n=12}^{\infty} \frac{(4x-9)^{2n}}{(n+1)!}$ has infinite rad. of conv.

$$\lim_{n\to\infty} \frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n}$$

$$= \lim_{n\to\infty} \left|(n+1)(x-2)\right| > 1 \text{ if } x-2 \neq 0.$$

$$\lim_{n\to\infty}\left|\frac{\frac{(4x-9)^{2n+2}}{(n+2)!}}{(4x-9)^{2n}/(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{1}{n+2}(4x-9)^2\right| = 0$$

for all x.

More on (3)

(3) says $< R$ or $> R$. What about $= R$?

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Examples:

3a) $\sum_{n=0}^{\infty}(3x-2)^n$ has rad. of conv. $1/3$. Converges on $(1/3, 1)$.
Diverges if $|3x-2| \geq 1$, on $(-\infty, 1/3]$ and $[1, \infty)$

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Diverges if $|3x-2| \geq 1$, on $(-\infty, 1/3]$ and $[1, \infty)$

3b) $\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n}$ has rad. of conv. $1/3$. Converges if $(3x-2) = -1$, diverges if $3x-2 \geq 1$: converges on $[1/3, 1)$, diverges otherwise.

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3b) $\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n}$ has rad. of conv. $1/3$. Converges if $(3x-2) = -1$, diverges if $3x-2 \geq 1$: converges on $[1/3, 1)$, diverges otherwise.

3c) $\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n^2}$ converges also if $3x-2 = 1$, so converges on $[1/3, 1]$ diverges otherwise.

Example

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$$f(x) = 2x\left(\frac{1}{1-(-x^2)}\right) = 2x\sum_{n=0}^{\infty}(-x^2)^n = 2x - 2x^3 + 2x^5 \mp \ldots$$

Or,

$$f(x) = \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1}$$

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Rad. of conv. is when

$$\left|\frac{2(-1)^{n+1}x^{2(n+1)+1}}{2(-1)^n x^{2n+1}}\right| = |x^2| < 1,$$

or $|x| < 1$.

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$$f(x) = x^3 \frac{1}{4(1+x/4)} = \frac{x^3}{4}\left(\frac{1}{1-(-x/4)}\right) = \frac{x^3}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{4}\right)^n$$

or

$$f(x) = \sum_{n=0}^{\infty}(-1)^n 4^{-n+1} x^{n+3} = \sum_{m=3}^{\infty}(-1)^{-m+3}4^{-m+2}x^m$$

(replacing $m = n + 3$)

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$$f(x) = x^3 \frac{1}{4(1 + x/4)} = \frac{x^3}{4} \left(\frac{1}{1 - (-x/4)} \right) = \frac{x^3}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{4} \right)^n$$

or

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(replacing $m = n + 3$)
Rad. of conv. is when $|4^{-1} x| < 1$, or $|x| < 4$. $R = 4$.

Differentiation and Integration

Theorem:
Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series centered at a, with radius of convergence $R > 0$ (converges on $(a-R, a+R)$). Then f is differentiable and integrable on $(a-R, a+R)$, and you can differentiate or integrate term by term.

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$$\begin{aligned}\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n &= \frac{d}{dx}(c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \\ &= c_1 + 2c_2(x-a) + 3c_3(x-a)^3 + \ldots \\ &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.\end{aligned}$$

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&= c_1 + 2c_2(x-a) + 3c_3(x-a)^3 + \ldots \\
&= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.
\end{aligned}$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$$

$+C$ necessary here!

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f is a derivative:

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$$\frac{d}{dx}\left(\frac{1}{1+2x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-2n)(-2x)^{n-1}.$$

Reindex

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$$\rightsquigarrow -2\sum_{n=0}^{\infty}(n+1)(-2x)^n = -2f(x)$$

Rad. of conv. Ratio test $\rightsquigarrow |2x| < 1$, so $R = 1/2$.

Example

Find a power series representing $\ln(1+3x)$ centered at $a=0$. What is the radius of convergence?

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n x^n\right)$$

$$= \sum_{n=1}^{\infty} \underline{n c_n x^{n-1}}$$

$$\int \sum_{n=0}^{\infty} c_n x^n \, dx = \sum_{n=0}^{\infty} \int c_n x^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{n} x^{n+1} + \underline{C}$$

$$\frac{d}{dx}(f+g) = \underbrace{\frac{df}{dx}}_{f'} + \underbrace{\frac{dg}{dx}}_{g'}$$

$$\int f+g\,dx = \int f\,dx + \int g\,dx$$

$$f(x) = \ln(1+3x)$$

$$f'(x) = \frac{3}{1+3x} \nearrow$$

This form we can find the power series for.

$$f(x) = \int f'(x)\,dx = \int \frac{3}{1+3x}\,dx$$

$$= \int \sum_{n=0}^{\infty} c_n x^n\,dx$$

$$= \sum_{n=0}^{\infty} \int c_n x^n\,dx + C$$

Example

Find a power series representing $\ln(1+3x)$ centered at $a=0$.
What is the radius of convergence?

$$\ln(1+3x) = 3\int \left(\frac{1}{1+3x}\right) dx + C$$

we can write as a power

$$\frac{1}{1-(-3x)} = \sum_{n=0}^{\infty} (-3x)^n$$

$$\ln(1+3x) = 3\int \sum_{n=0}^{\infty} (-3x)^n \, dx + C$$

$$= 3\sum_{n=0}^{\infty} \int (-3x)^n \, dx$$

$$= 3\sum_{n=0}^{\infty}\frac{(-3)^n}{n+1}x^{n+1} + C$$

To find C plug in $x=0$.

$$\ln(1+0) = C$$

$$\ln(1) = 0 = C.$$

$$= \frac{3}{-3}\sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{n+1}x^{n+1}$$

$$= \boxed{-\sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{n+1}x^{n+1}}$$

$$= \boxed{-\sum_{k=1}^{\infty}\frac{(-3)^k}{k}x^k}$$

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$$\ln(1+3x) = 3\int \left(\frac{1}{1+3x}\right) dx + C$$

$$\ln(1+3x) = 3\int \left(\sum_{n=0}^{\infty}(-3x)^n\right) dx$$

$$= 3\sum_{n=0}^{\infty}\frac{(-3x)^{n+1}}{(-3)(n+1)} + C$$

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$$= 3\sum_{n=0}^{\infty} \frac{(-3x)^{n+1}}{(-3)(n+1)} + C$$

How do you find C? Plug in $x=0$: $\ln(1+3\cdot 0) = \ln(1) = 0$. In series:

$$\sum_{n=0}^{\infty} \frac{(-3\cdot 0)^{n+1}}{n+1} + C = C$$

Find a power series representing $\ln\left(\frac{1+x}{1-x}\right)$.

$$\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B).$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x)^n\,dx$$

$$= \sum_{n=0}^{\infty} \int (-x)^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} -\left(\frac{(-1)^{n+1} x^{n+1}}{n+1} \right)$$

$$(-1)^n = (-1)^2 (-1)^n = (-1)(-1)^{n+1}$$

$$-\ln(1-x)^1 = \int \frac{1}{1-x} \, dx \qquad \frac{d}{dx}\left[- \quad (\ln(1-x)) \right]$$

$$= -\frac{1}{1-x} \cdot (-1)$$

$$= \int \sum_{n=0}^{\infty} x^n \, dx \qquad = \frac{1}{1-x}$$

$$= \sum_{n=0}^{\infty} \int x^n dx$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} -\left(\frac{(-1)^{n+1} x^{n+1}}{n+1} \right) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

Plug in $x=0$. $\ln\left(\frac{1}{1}\right) = 0 = 0 + 0 + C \Rightarrow C = 0$.

Exercise

Find a power series representing $\ln\left(\frac{1+x}{1-x}\right)$.

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \int \left(\sum_{n=0}^{\infty}(-x)^n + \sum_{n=0}^{\infty} x^n\right) dx \\
&= \sum_{n=0}^{\infty}\left(-\frac{(-x)^{n+1}}{n+1} + \frac{x^{n+1}}{n+1}\right) + C \\
&= \sum_{n \text{ even}} 2\frac{x^{n+1}}{n+1}. = \sum_{k=0}^{\infty} 2\left(\frac{x^{2k+1}}{2k+1}\right)
\end{aligned}$$

even $2k$

odd $2k+1$

$$\underline{n=0} \quad -\frac{(-x)^1}{1} + \frac{x^1}{1} = \frac{x}{1} + \frac{x}{1} = \frac{2x}{1}.$$

$$\underline{n=1} \quad -\frac{(-x)^2}{2} + \frac{x^2}{2} = \underline{0}$$

$$\underline{n=2} \quad -\frac{(-x)^3}{3} + \frac{x^3}{3} = \frac{2x^3}{3}.$$

$$\underline{n=3} \quad -\frac{(-x)^4}{4} + \frac{x^4}{4} = 0$$

Taylor series motivation

power series centered at $x=0$

Let $f(x) = 5 - 4x + 2x^2 + x^3. + 0x^4 + 0x^5 + 0x^6 + \ldots$

$f'(x) = -4 + 4x + 3x^2$

$f'(0) = -4$

$f'(1) = 3$

$f''(x) = 4 + 6x$

$f''(0) = 4$

$f''(1) = 10$

$f'''(x) = 6$

$f'''(0) = 6$

$f'''(1) = 6$

$f(1) = 5 - 4 + 2 + 1 = 4$

Let $f(x) = 5 - 4x + 2x^2 + x^3$.

$$f(0) = 5,\ \ f'(0) = (-4 + 4x + 3x^2)|_{x=0} = -4,$$

$$f''(0) = (4 + 6x)|_{x=0} = 4,\ \ f'''(0) = 6.$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3$$

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Let $f(x) = 5 - 4x + 2x^2 + x^3$.

$$f(0) = 5,\ \ f'(0) = (-4 + 4x + 3x^2)|_{x=0} = -4,$$

$$f''(0) = (4 + 6x)|_{x=0} = 4,\ \ f'''(0) = 6.$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

Different center?

Now let $f(x) = 5 - 4x + 2x^2 + x^3$. Find a power series representing f centered at $a = 1$.

$f(x) = c_0 + c_1(x-a) + \dots$

rewrite x as $(x-1)+1$ and expand the terms.

$$f(x) = 5 - 4((x-1)+1) + 2((x-1)+1)^2 + ((x-1)+1)^3$$

$$= 5 - 4(x-1) - 4 + 2(x-1)^2 + 4(x-1) + 2$$

$$+ (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$$

$\sin(1/n) \approx 1/n$

Let $f(x) = \sin x$. Find a power series centered at $a = 0$ representing f. What is the radius of convergence? $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, ... so

$$f^{(n)}(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^k, & n \text{ even}, n = 2k. \end{cases}$$

$$f(x) = 0 + x + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5 \dots$$
$$= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

Exercise

Let $f(x) = \sin x$. Find a power series centered at $a = 0$ representing f. What is the radius of convergence? $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, ... so

$$f^{(n)}(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^k, & n \text{ even}, n = 2k. \end{cases}$$

$$\begin{aligned} f(x) &= 0 + x + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5 \dots \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} \end{aligned}$$

Does the derivative of this series equal $\cos x$?

Example

Let $f(x) = e^{2x}$. What is $f^{(10)}(2)$?

If we have a power series we can find derivatives easily

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doesn't quite work, since coefficients have information about f at $a = 0$.

$$e^{2x} = e^{2(x-2+2)} = e^{2(x-2)+4} = e^4 e^{2(x-2)},$$

so

$$f(x) = e^4 \sum_{n=0}^{\infty} \frac{(2(x-2))^n}{n!}$$

then $c_n = e^4 \frac{2^n}{n!} = \frac{f^{(n)}(2)}{n!}$, so $f^{(10)}(2) = e^4 2^n$. $= e^4 2^{10}$

Exercise

Let $f(x) = \sin(3x)$. Find $f^{(10)}(1)$ Using a Taylor Series

Exercise

Let $f(x) = \sin(3x)$. Find $f^{(10)}(1)$.

$$\begin{aligned}\sin(3x) &= \sin(3(x-1+1)) = \sin((3(x-1)+3) \\ &= \sin(3(x-1))\cos 3 + \sin 3\cos(3(x-1)),\end{aligned}$$

so

$$f(x) = \cos(3)\sum_{n=0}^{\infty}(-1)^n\frac{(3(x-1))^{2n+1}}{(2n+1)!} + \sin(3)\sum_{n=0}^{\infty}(-1)^n\frac{(3(x-1))^{2n}}{(2n)!}$$

If $k = 2n+1$ is odd, then $c_k = (-1)^n\cos(3)\frac{3^{2n+1}}{(2n+1)!}$, and if $k = 2n$ is even, $c_k = (-1)^n\sin(3)\frac{3^{2n}}{(2n)!}$.
$k = 10 \rightsquigarrow n = 5$, so

$$f^{(10)}(1) = \sin(3)3^{10}$$

Taylor polynomials

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Then partial sums are

$$T_0 = f(a)$$

$$T_1 = f(a) + f'(a)(x-a),$$

In general,

$$T_n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}$$

is a polynomial of degree n.

Convergence and remainders

Let $f(x)$ be a function having derivatives of all orders on an interval $(a - R, a + R)$, and let T_n be the nth Taylor polynomial. The *nth remainder* R_n is $R_n(x) = f(x) - T_n(x)$.

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Theorem:
1) If $R_n(x) \to 0$ as $n \to \infty$ for every x in $(a - R, a + R)$, then $f(x)$ is equal to its Taylor series on $(a - R, a + R)$. (f is then called *analytic*)

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2) For $a - R < x < a + R$, The remainder $R_n(x)$ satisfies

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - a)^{n+1}$$

with z somewhere between x and a. (Important! We don't know what z is.)

Example

$f(x) = e^x$. For every n, we have $f^{(n)}(x) = e^x$. The series $\sum_0^\infty \frac{x^n}{n!}$ has infinite radius of convergence.

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Question: does it actually converge to e^x?
Check: does $R_n(x) \to 0$?

$$T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}.$$

Example continued

Fix an x_0. Need to show $R_n(x_0) \to 0$ for this x_0.

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$$R_n(x_0) = \frac{f^{(n+1)}(z)}{(n+1)!} x_0^{n+1},$$

for some z between x_0 and 0.

Example continued

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For a fixed x_0, $|z| \leq |x_0|$, so e^z is bounded by $e^{|x_0|}$.

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$$R_n(x_0) = \frac{e^z}{(n+1)!} x_0^{n+1}.$$

For a fixed x_0, $|z| \leq |x_0|$, so e^z is bounded by $e^{|x_0|}$. Since x_0 is frozen,

$$\frac{e^{|x_0|} x_0^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$ (why?).

Exercise

Let $f(x) = \sin(x)$. Prove that the Taylor series for $\sin(x)$ converges to $\sin(x)$.

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Look at remainders:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}.$$

$f^{(n)}(z) = \pm\sin(z),$ or $\pm\cos(z)$, which are all bounded by 1.
Hence

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

Examples

Taylor series we know:

1. $$\frac{1}{1-x} = \sum_{0}^{\infty} x^n, \;\; |x| < 1,$$

2. $$e^x \text{ has T.S. } \sum_{0}^{\infty} \frac{x^n}{n!} \;\; \text{any } x$$

3. $$\sin x \text{ has T.S. } \sum_{0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;\; \text{any } x$$

4. $$\cos x \text{ has T.S. } \sum_{0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \;\; \text{any } x$$

Examples without T.S.

Example $f(x) = |x - 2|$.

Examples without T.S.

Example $f(x) = |x - 2|$. f not differentiable at $x = 2$ (why?)

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More interesting: Let

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$$

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That means f cannot have a Taylor series centered at $a = 2$.
What about other a?
More interesting: Let

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$$

Can show that f has derivatives of all orders, and $f^{(n)}(0) = 0$ for all $n \geq 0$. That means the T.S. for this f is

$$\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$$

which is not $f(x)$.

Application

Let $f(x) = \frac{x}{(1-2x)^2}$. Find a power series for f centered at $a = 1$ and use it to compute $f^{(10)}(1)$.

In this case $f^{(10)}(x)$ is complicated since we got a bunch of quotient rules.

Application

Let $f(x) = \frac{x}{(1-2x)^2}$. Find a power series for f centered at $a = 1$ and use it to compute $f^{(10)}(1)$.
Let $g(x) = \frac{1}{1-2x}$, $g'(x) = \frac{2}{(1-2x)^2}$, so $f(x) = \frac{1}{2}xg'(x)$.

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$$g(x) = \frac{1}{1-2(x-1+1)} = -\frac{1}{1+2(x-1)} = -\sum_0^\infty (-2(x-1))^n$$

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(where does it converge?)

$$\frac{1}{2}xg'(x) = \frac{1}{2}((x-1)+1)(-\sum_0^\infty n(-2(x-1))^{n-1}(-2))$$

$$= \sum_0^\infty n(-2)^{n-1}(x-1)^n + \sum_0^\infty n(-2)^{n-1}(x-1)^{n-1}$$

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$$= \sum_0^\infty n(-2)^{n-1}(x-1)^n + \sum_0^\infty n(-2)^{n-1}(x-1)^{n-1}$$

Coefficient of $(x-1)^{10}$ is $\frac{f^{(10)}(1)}{10!} = (10(-2)^9 + 11(-2)^{10})$.

Exercise

Let $f(x) = \frac{x+3}{1-2x}$. Find a power series for f centered at $a = -1$ and use it to compute $f^{(100)}(-1)$.

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$$f(x) = ((x+1)+2)\frac{1}{3-2(x+1)}$$

$$= \frac{1}{3}\sum_{0}^{\infty}\left(\frac{2}{3}\right)^n (x+1)^{n+1} + \frac{2}{3}\sum_{0}^{\infty}\left(\frac{2}{3}\right)^n (x+1)^n$$

Exercise

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$$f(x) = ((x+1)+2)\frac{1}{3-2(x+1)}$$
$$= \frac{1}{3}\sum_{0}^{\infty}\left(\frac{2}{3}\right)^{n}(x+1)^{n+1} + \frac{2}{3}\sum_{0}^{\infty}\left(\frac{2}{3}\right)^{n}(x+1)^{n}$$

Coefficient of $(x+1)^{100}$ is

$$\frac{f^{(100)}(-1)}{100!} = \left(\frac{1}{3}\left(\frac{2}{3}\right)^{99} + \frac{2}{3}\left(\frac{2}{3}\right)^{100}\right)$$

Exercise

Let $f(x) = x\cos(x)$. Find a Taylor series for f. Prove that the Taylor series converges to f. You may use the Taylor series for cosine:

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

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Taylor series for f:

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}$$

Remainder: Fix x_0. For each N, there is a z between 0 and x_0 such that

$$R_N(x_0) = \frac{f^{(N+1)}(z)}{(N+1)!}(x_0)^{N+1}$$

Continued..

Need to understand derivatives of f.

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For every k, $f^{(k)}$ is a polynomial of degree at most 1 times $\sin x$ and/or $\cos x$. The largest coefficients in these polynomials $\sim 2^{N+1}$.

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For every k, $f^{(k)}$ is a polynomial of degree at most 1 times $\sin x$ and/or $\cos x$. The largest coefficients in these polynomials $\sim 2^{N+1}$.

$$\rightsquigarrow |R_N(x_0)| \lesssim \frac{2^{N+1}}{(N+1)!}|x_0|^{N+1} \to 0$$

as $N \to \infty$.

Exercise

Let $f(x) = (1 + x^4)\cos(2x)$. Find $f^{(29)}(0)$. Find $f^{(30)}(0)$.

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Taylor series centered at $a = 0$:

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{(2n)!}$$

Exercise

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Taylor series centered at $a = 0$:

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+4}}{(2n)!}$$

Coefficient of x^{29} is 0, since f is an even function! Coefficient of x^{30} is ... $30 = 2 \cdot 15$, so in first sum $n = 15$, in second sum $n = 13$, so coefficient of x^{30} is

$$\frac{f^{(30)}(0)}{30!} = (-1)\frac{1}{30!} + (-1)\frac{1}{26!}$$

Applications of Taylor series

1) Approximating functions

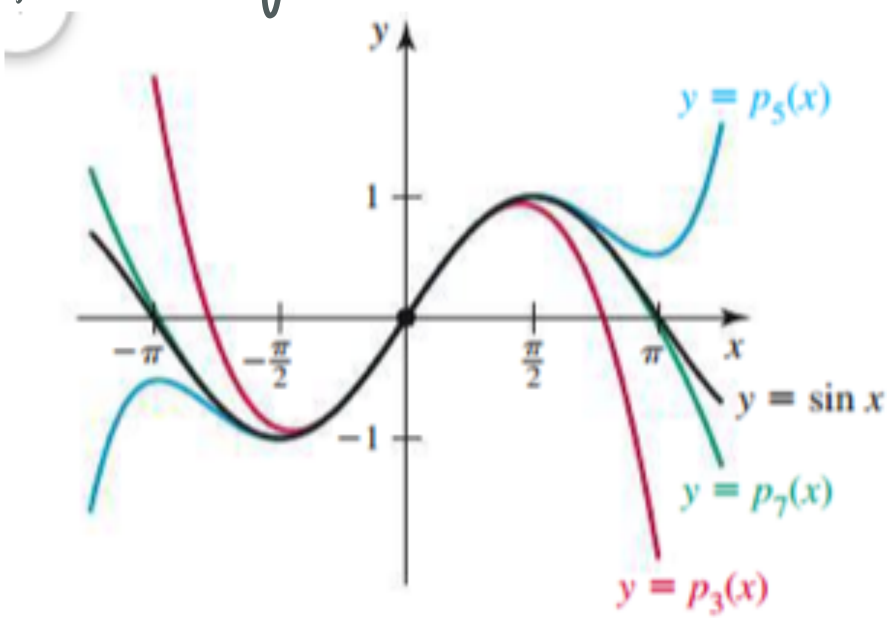

Figure 11.7

2) Estimating Values

3) Solving limits

4) Solving differential equations

5) Estimating integrals

6) Finding high order derivatives

7) Finding values of Sums

Table 11.5

$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k$, for $\lvert x\rvert < 1$
$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k$, for $\lvert x\rvert < 1$
$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, for $\lvert x\rvert < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, for $\lvert x\rvert < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, for $\lvert x\rvert < \infty$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$, for $-1 < x \le 1$
$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}$, for $-1 \le x < 1$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, for $\lvert x\rvert \le 1$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$, for $\lvert x\rvert < \infty$
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$, for $\lvert x\rvert < \infty$
$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k$, for $\lvert x\rvert < 1$ and $\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$, $\binom{p}{0} = 1$

2) We covered and did examples showing convergence

2) Estimate the remainder of $\sin(0.3)$ with 3^{rd} order Taylor Series approximation.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \underbrace{x - \frac{x^3}{3!}}_{\text{4th order term}} + \cdots$$

$$\sin(0.3) = 0.3 - \frac{0.3^3}{3!} = 0.2955$$

Actual is $\approx 0.29552020666\ldots$

Estimate on the error.

$$R_3 \leq \left| \frac{f^{(4)}(c)}{4!} (0.3)^4 \right| \leq \left| \frac{0.3^4}{4!} \right|$$

$$= 0.000375.$$

Exercise: Estimate the error in approximating $e^{0.25}$ with a 4th order Taylor series centered at $a=0$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{0.25} \approx 1 + 0.25 + \frac{0.25^2}{2} + \frac{0.25^3}{3!} + \frac{0.25^4}{4!}$$

$$\approx 1.28402$$

$$R_4 \leq \left| \frac{f^5(c)}{5!} 0.25^5 \right| \leq \left| \frac{e^{0.25}}{5!} 0.25 \right|$$

c is between 0 and 0.25 $\leq \left| \frac{2 \cdot 0.25}{5!} \right|$

$$= 0.00416\bar{6}$$

Actual is 1.2840254...

So we are within this range.

3) Compute $\lim\limits_{x \to 0} \dfrac{\sin(1/x)}{x}$

$$\frac{\sin(1/x)}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

So

$$\lim_{x \to 0} \frac{\sin(1/x)}{x} = \lim_{x \to 0} 1 - \frac{x^2}{3} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$= 1.$$

Exercise: Compute $\displaystyle\lim_{x\to 0} \frac{2\sin x - \tan^{-1}x - x}{2x^5}$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

This is because $\tan^{-1}(x) = \int \frac{1}{1-(-x)^2}\,dx$

$$\frac{2\sin x - \tan^{-1}(x) - x}{2x^5}$$

$$= \frac{1}{2x^5}\left(2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} - x\right)$$

$$= \frac{1}{2x^5}\left(2\left(x - \frac{x^3}{6} + \frac{x^5}{5!} - \dots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) - x\right)$$

$$= \frac{1}{2x^5}\left(\left(\frac{2x^5}{5!} - \frac{x^5}{5}\right) + \overbrace{\left(-\frac{2x^7}{7!} + \frac{x^7}{7} + \dots\right)}^{\text{higher order terms}}\right)$$

So $\lim\limits_{x \to 0} \dfrac{2\sin x - \tan^{-1}(x) - x}{2x^5}$

$= \dfrac{1}{5!} - \dfrac{1}{10} = \boxed{-\dfrac{11}{120}}$

4) Find the solution to

$y' - 4y + 2 = 0$ with $y(0) = 4$

$y = \sum\limits_{n=0}^{\infty} C_n x^n$

$y' = \sum\limits_{n=0}^{\infty} C_n n x^{n-1}$

so,

$\sum\limits_{n=0}^{\infty} C_n n x^{n-1} - 4\sum\limits_{n=0}^{\infty} C_n x^n + 2 = 0$

c $\left[C_1 + C_2 2x + C_3 3x^2 + C_4 4x^3 + \cdots\right]$

$- \left[4C_0 + 4C_1 x + 4C_2 x^2 + 4C_3 x^3 + \cdots\right]$

$+2=0$

Not $y(0)=4$ so $C_0=4$

So, $C_1-4C_0+2=0 \qquad C_1=14$

$2C_2-4C_1=0 \qquad C_2=28$

$3C_3-4C_2=0 \qquad C_3=\frac{4\cdot 28}{3}$

$\vdots$

$nC_n-4C_{n-1}=0 \qquad C_n=\frac{4C_{n-1}}{n}$

So in general. $y=4+14\sum_{n=1}^{\infty}\frac{4^{n-1}}{n!}x^n$

$$y=4+\frac{7}{2}\sum_{n=1}^{\infty}\frac{4^n x^n}{n!}$$

$$=4+\frac{7}{2}\sum_{n=1}^{\infty}\frac{4^n x^n}{n!}$$

$$= \left(4 - \frac{7}{2}\right) + \frac{7}{2} \sum_{n=0}^{\infty} \frac{4^n x^n}{n!}$$

$$= \frac{1}{2} + \frac{7}{2} \sum_{n=0}^{\infty} \frac{(4x)^n}{n!}$$

$$= \frac{1}{2} + \frac{7}{2} e^{4x}.$$

Check

$$y' = 14 e^{4x}, \quad -4y = \left(\frac{-4}{2} - 14 e^{-4x}\right)$$

$$= -2 - 14 e^{-4x}$$

so $y' - 4y + 2 = 0$? ✓

$$14e^{4x} - 2 - 14e^{4x} + 2 = 0 \checkmark$$

$$y(0) = \frac{1}{2} + \frac{7}{2} e^{4 \cdot 0} = \frac{8}{2} = 4 \checkmark$$

5) Use a Taylor Series to approximate the following definite integrals. Retain as many terms as necessary to

ensure the error is less than 10^{-3}.

(a) $\int_0^{1/2} e^{-x^2}\, dx$

Remark: This is hard to explicitly integrate so we'd have to use numerical methods.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int_0^{1/2} e^{-x^2}\, dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\, dx$$

$$= \sum_{n=0}^{\infty} \int_0^{1/2} \frac{(-1)^n x^{2n}}{n!}\, dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+1}}{(2n+1)\, n!} \right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+1}}{(2n+1)\, n!}$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^3 \Big/ 3\cdot 1! + \frac{\left(\frac{1}{2}\right)^5}{5\cdot 2!} - \frac{\left(\frac{1}{2}\right)^7}{7\cdot 3!}$$

This is an alternating series so $R_n \leq |a_{n+1}|$

so we need $\frac{\left(\frac{1}{2}\right)^{2n+1}}{(2n+1)n!} < 0.001$

Just guess and check.

$|a_2| = \frac{1}{320} \not< \frac{1}{100}$

$|a_3| = \frac{1}{5376} < \frac{1}{1000}$

$\int_0^{1/2} e^{-x^2}\,dx \approx \frac{1}{2} - \frac{1}{24} + \frac{1}{320} = \frac{443}{960} \approx \underline{0.461}$

Using a calculator

$\int_0^{1/2} e^{-x^2}\,dx = 0.\underbrace{461}281$

First 3 digits match.

① Exercise: $\int_0^{1/2} \tan^{-1}(x)\,dx$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\int_0^{1/2} \tan^{-1}(x) = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \int_0^{1/2} \frac{(-1)^n x^{2n+1}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+2}}{(2n+2)(2n+1)}\right]_0^{1/2}$$

$$= \frac{(1/2)^2}{2} - \frac{(1/2)^4}{4\cdot 3} + \frac{(1/2)^8}{6\cdot 5} - \cdots$$

$R_n \leq |a_{n+1}|$

Need to find $\frac{(1/2)^{2n}}{(2n+2)(2n+1)} < \frac{1}{100}$

Guess and check

$a_1 = \frac{1}{192} \nless \frac{1}{1000}$

$a_2 = \frac{1}{1920} < \frac{1}{1000}$ ✓

so we need the first terms

$\int_0^{1/2} \tan^{-1}(x)\,dx = \frac{1}{8} - \frac{1}{192} = \frac{23}{192} = \underline{0.119}792\ldots$

In this case we can solve the integral to get $\int_0^{1/2} \tan^{-1}(x)\,dx = \frac{1}{2}\left[\tan^{-1}\left(\frac{1}{2}\right) - \ln\left(\frac{5}{4}\right)\right]$

$= 0.\underline{120}252\ldots$

within .001 of the real answer.

6) We already did.

7) Find the value of $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

Recall that $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

so, $\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Note: This gives us a formula to estimate π.

$$\underline{4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}} = \pi.$$

This is an alternating series so we can estimate the error. (This algorithm isn't used because convergence is slow.) Roughly need 200 terms to be within $\frac{1}{100}^{th}$ of the correct value)

Example: Basel Problem

The power series $\frac{\sin(x)}{x}$ is used to show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (Famous problem solved in 1734 by Euler)