

# Directional Derivatives

June 24, 2020

- Big Picture:** 1) Directional derivatives are generalizations of partial derivatives to “different directions”.
- 2) The gradient can be used to find *steepest ascent*.

# Directional derivative: Motivation

**Recall:** In 1-d, the derivative (when it is defined) is

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This is  $\langle a, b \rangle +$  step of size  $h$  in  $\langle 1, 0 \rangle$  direction.

# Rewrite $x$ derivative

*Rewrite in vector notation:*

$$\begin{aligned} f_x(a, b) &= \frac{f(a + h, b) - f(a, b)}{h} \\ &= \frac{f(\langle a, b \rangle + h \langle 1, 0 \rangle) - f(\langle a, b \rangle)}{h} \end{aligned}$$

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$$D_{\mathbf{u}}f(a, b) = \lim_{h \rightarrow 0} \frac{f(\langle a, b \rangle + h\mathbf{u}) - f(\langle a, b \rangle)}{h}.$$

# Vary $x$ direction, or $\langle 1, 0 \rangle$ direction

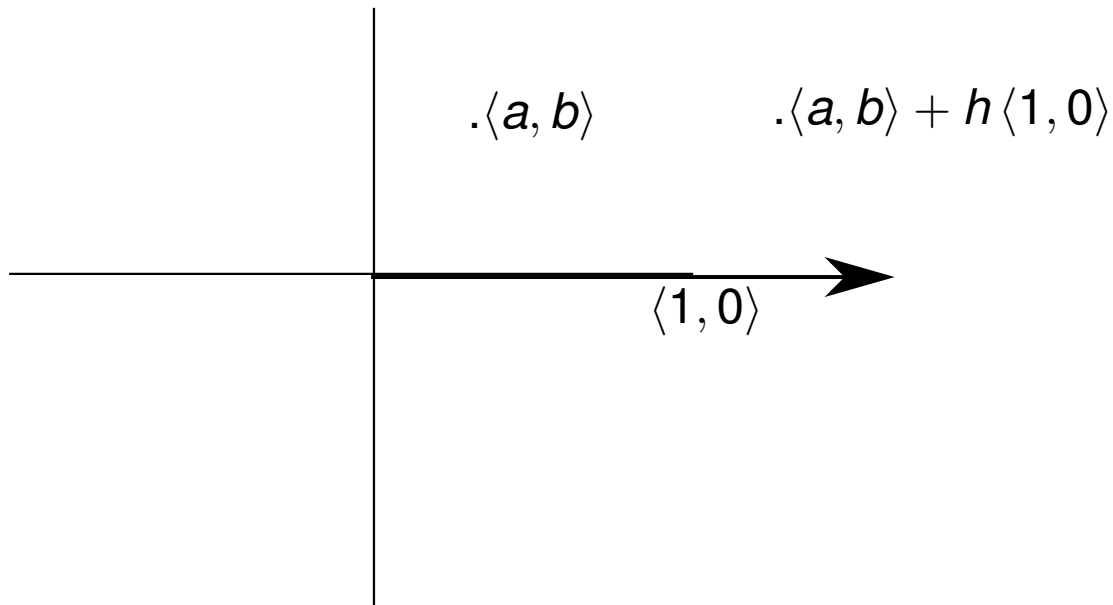


Figure: Stepping by  $h$  in the  $x$  direction.

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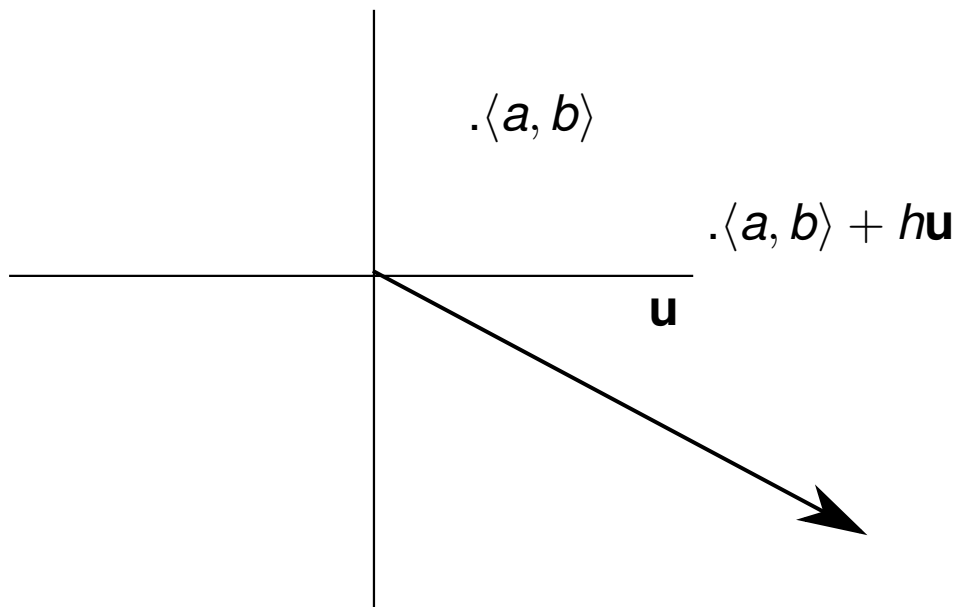


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First part is a  $y$  derivative, second part is an  $x$  derivative. Chain rule  $\rightsquigarrow$

$$\lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = f_x(a, b)u_1 + f_y(a, b)u_2.$$

# Example

Let  $f(x, y) = x^2 + 100y^2$ . Find  $D_{\mathbf{u}}f(-1, 0)$  for the vectors  $\mathbf{u} = \langle 1, 0 \rangle$ ,  $\mathbf{u} = \langle 0, 1 \rangle$ , and  $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ .

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$f_x = 2x$ ,  $f_y = 200y$ . Plugging in  $(-1, 0)$  gives  $f_x(-1, 0) = -2$ ,  
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35 → 52

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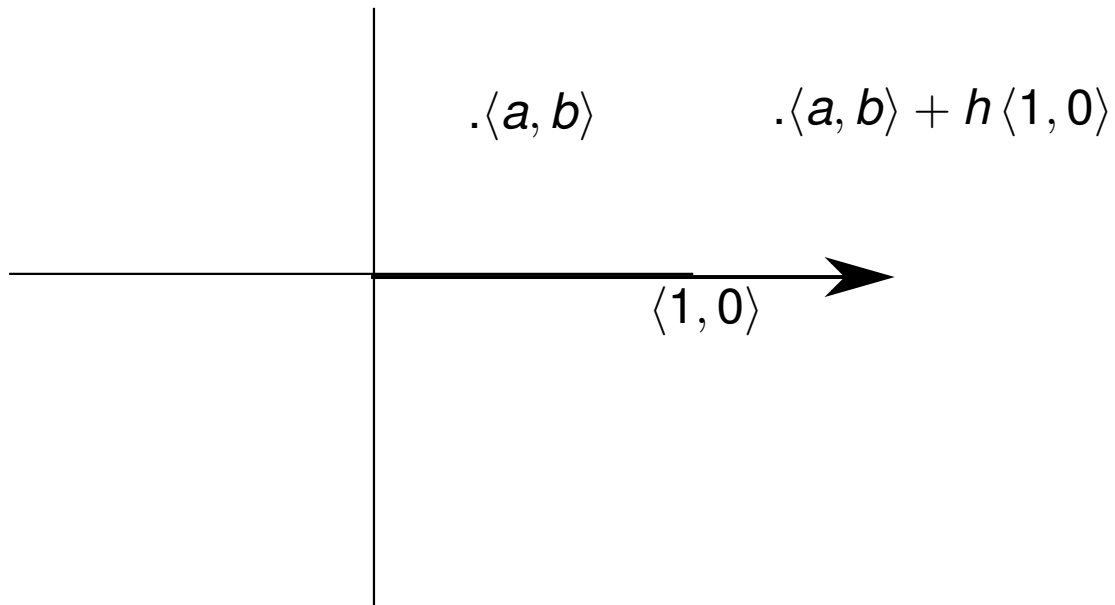


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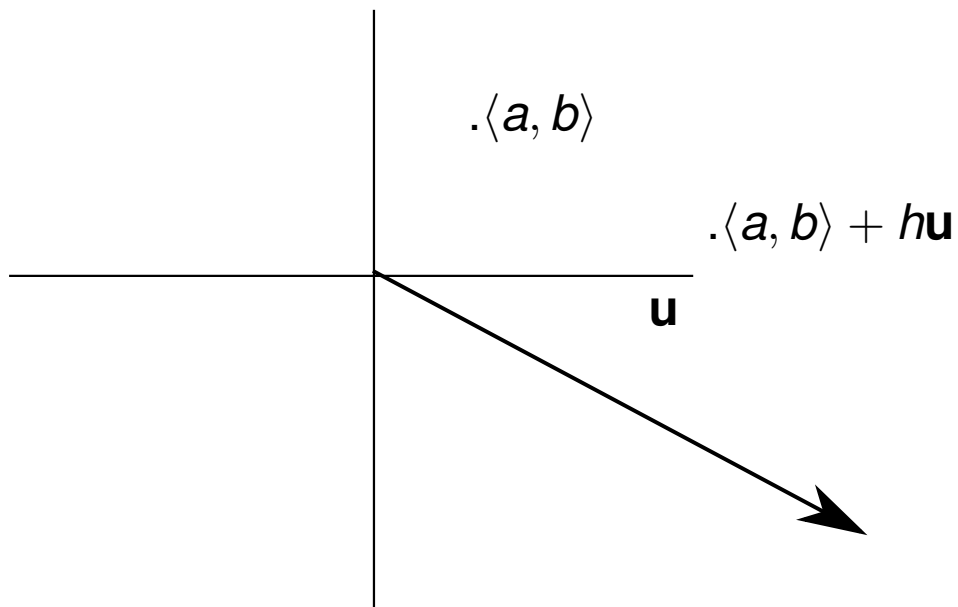


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# Computing the directional derivative

Look at the difference quotient:

$$\begin{aligned} & \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \\ &= \frac{f(a + hu_1, b + hu_2) - f(a + hu_1, b) + f(a + hu_1, b) - f(a, b)}{h} \end{aligned}$$

First part is a  $y$  derivative, second part is an  $x$  derivative. Chain rule  $\rightsquigarrow$

$$\lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = f_x(a, b)u_1 + f_y(a, b)u_2.$$

# Example

Let  $f(x, y) = x^2 + 100y^2$ . Find  $D_{\mathbf{u}}f(-1, 0)$  for the vectors  $\mathbf{u} = \langle 1, 0 \rangle$ ,  $\mathbf{u} = \langle 0, 1 \rangle$ , and  $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ .

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$f_x = 2x$ ,  $f_y = 200y$ . Plugging in  $(-1, 0)$  gives  $f_x(-1, 0) = -2$ ,  $f_y(-1, 0) = 0$ . Then

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 $f_x(1, 1) = 2$  and  $f_y(1, 1) = 200$ . Then

$$D_{\langle 1, 0 \rangle} f(1, 1) = 2, \quad D_{\langle 0, 1 \rangle} f(1, 1) = 200,$$

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# The gradient

**Definition:** Let  $D$  be a domain in  $\mathbb{R}^2$ ,  $f(x, y) : D \rightarrow \mathbb{R}$  differentiable. Let  $(a, b)$  be a point in  $D$ . Then the *gradient* of  $f$  at  $(a, b)$  is the *vector*

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# Application: steepest ascent/descent

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Any unit vector can be written  $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ , so

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which has max 2 at  $\theta = 0$ , and min  $-2$  at  $\theta = \pi$ .

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$$g(3\pi/4) = \sqrt{2} + \sqrt{2}.$$

# Exercise

Let  $f = xy$ . Find  $\nabla f(\sqrt{3}, -1)$ . Find  $D_{\langle 0, -1 \rangle} f(\sqrt{3}, -1)$ . Find  $\mathbf{u}$  which maximizes  $D_{\mathbf{u}} f(\sqrt{3}, -1)$ .

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If  $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ , then

$$D_{\mathbf{u}} f(\sqrt{3}, -1) = -\cos \theta + \sqrt{3} \sin \theta$$

is maximized when  $\sin \theta + \sqrt{3} \cos \theta = 0$ , or  $\tan \theta = -\sqrt{3}$ , so  $\theta = 2\pi/3$  or  $-\pi/3$ . Max is where  $\theta = 2\pi/3$ .