Homework Problems

This is a running list of the homework problems. Each homework assignment will appear on Gradescope.

1. Reduce the following numbers to the form $a + ib$:
   (i) $\left(\dfrac{2+i}{3-2i}\right)^2$
   (ii) $\dfrac{z-1}{z+1}$, where $z = x + iy$, with x and y real,
   (iii) $\dfrac{1}{z^2}$, where $z = x + iy$, with x and y real.
2. Depict $z_1 + z_2$ and $z_1 - z_2$ graphically.
   (i) $z_1 = 2i, z_2 = 2/3 - i$
   (ii) $z_1 = x + iy, z_2 = x - iy$
3. If z is a complex number such that $|z| = 1$, compute $|1 + z|^2 + |1 - z|^2$
4. Show that $|z_1 z_2| = |z_1||z_2|$
5. Use mathematical induction to show that $|z^n| = |z|^n$ for $n = 1, 2, \ldots$.
6. Show that
   (i) $\overline{\bar{z} + 3i} = z - 3i$
   (ii) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$
   (iii) $\overline{z^4} = \bar{z}^4$

Homework #1

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1. Reduce the following numbers to the form $a + ib$:

   (i) $\left(\dfrac{2+i}{3-2i}\right)^2$

   (ii) $\dfrac{z-1}{z+1}$, where $z = x + iy$, with x and y real,

   (iii) $\dfrac{1}{z^2}$, where $z = x + iy$, with x and y real.

(i)

$$\left(\frac{2+i}{3-2i}\right)^2 = \left(\frac{2+i}{3-2i}\cdot\frac{3+2i}{3+2i}\right)^2 = \left(\frac{6+4i+3i-2}{9+4}\right)^2$$

$$= \left(\frac{4+7i}{13}\right)^2 = \frac{16+56i-49}{169} = \boxed{\frac{-33}{169}+\frac{56}{196}i}$$

(ii)

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy}\cdot\frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x-1)(x+1)+(x+1)iy-(x-1)iy+y^2}{(x+1)^2+y^2}$$

$$= \frac{x^2+y^2-1+2iy}{x^2+2x+1+y^2} = \boxed{\frac{x^2+y^2-1}{x^2+2x+1+y^2}+\frac{2y}{x^2+2x+1+y^2}i}$$

(iii)

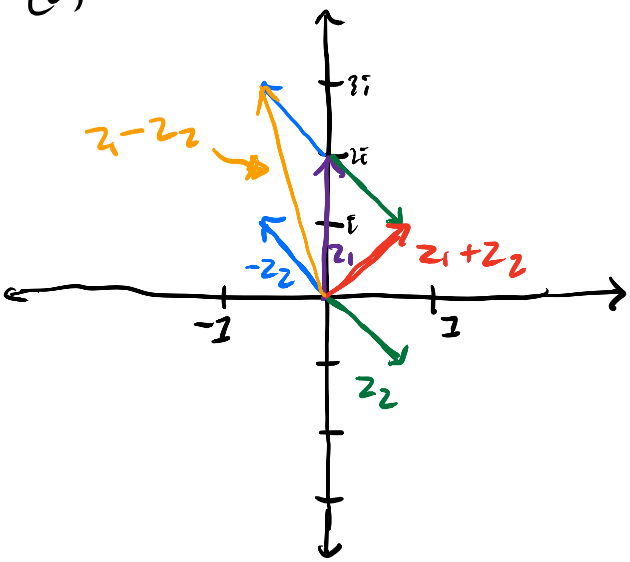
$$\frac{1}{z^2} = \frac{\bar{z}^2}{z^2\bar{z}^2} = \frac{(x-iy)^2}{(x^2+y^2)^2} = \boxed{\frac{x^2-y^2}{(x^2+y^2)^2}-\frac{2xy}{(x^2+y^2)^2}i}$$

2. Depict $z_1 + z_2$ and $z_1 - z_2$ graphically.

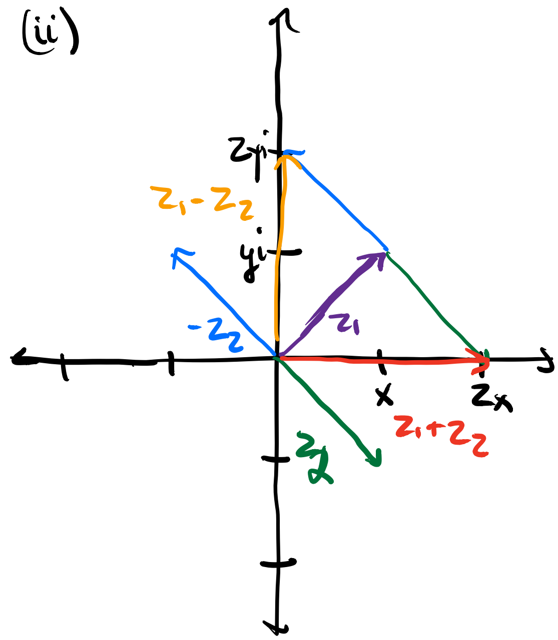
(i) $z_1 = 2i, z_2 = 2/3 - i$

(ii) $z_1 = x + iy, z_2 = x - iy$

(i)



(ii)



3. If z is a complex number such that $|z| = 1$, compute $|1 + z|^2 + |1 - z|^2$

$$|1+z|^2 + |1-z|^2 = (1+\bar{z})(1+z) + (1-z)(1-\bar{z})$$

$$= 1 + z + \bar{z} + z\bar{z} + 1 - z - \bar{z} + z\bar{z}$$

$$= 2 + 2z\bar{z} = 2 + 2|z|^2 = 2 + 2 = \boxed{4}$$

4. Show that $|z_1 z_2| = |z_1||z_2|$

Let $z_1 = a+bi$, $z_2 = c+di$ for $a,b,c,d \in \mathbb{R}$

$z_1 z_2 = (a+bi)(c+di) = [ac - bd] + [bc + ad]i$

So, $|z_1 z_2|^2 = (ac-bd)^2 + [bc+ad]^2$

$$= (ac)^2 - \cancel{2acbd} + (bd)^2 + bc^2 + \cancel{2ad}bc + ad^2$$

$$= (ac)^2 + (bd)^2 + (bc)^2 + (ad)^2$$

$$= (a^2+b^2)(c^2+d^2)$$

$$= |z_1|^2 |z_2|^2$$

Since $|z_1 z_2|, |z_1|, |z_2| \geq 0$ we have $|z_1 z_2| = |z_1||z_2|$

<u>Note</u>: Assuming we have $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ Then,

$$|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = z_1 z_2 \bar{z}_1 \bar{z}_2 = z_1 \bar{z}_1 z_2 \bar{z}_2$$

$$= |z_1|^2 |z_2|^2.$$

5. Use mathematical induction to show that $|z^n| = |z|^n$ for $n = 1, 2, \ldots$.

Note: You can directly prove this, but it says to prove using induction.

Base case: $n=1$ $\quad |z^1| = |z| = |z|^1$.

Inductive Step: Suppose for $n \in \mathbb{N}$ that $|z^n| = |z|^n$. Then,

$$|z^{n+1}| = |z^n z| \underbrace{=}_{\text{by problem 4}} |z^n||z| \underbrace{=}_{\text{by inductive hypothesis}} |z|^n |z| = |z|^{n+1}.$$

$\blacksquare$

6. Show that

(i) $\overline{\bar{z} + 3i} = z - 3i$

(ii) $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$

(iii) $\overline{z^4} = \bar{z}^4$

Let $z = x + iy$ for $x, y \in \mathbb{R}$

(i) $\overline{\bar{z} + 3i} = \overline{(x - iy) + 3i} = \overline{x + (3-y)i} = x - (3-y)i$

$= x + iy - 3i = z - 3i$ ∎

(ii) $|(2\bar{z} + 5)(\sqrt{2} - i)| = |2\bar{z} + 5|\,|\sqrt{2} - i|$ by problem 4

$= \sqrt{3}\,|2(x - iy) + 5|$

$= \sqrt{3}\,|2x + 5 - 2iy|$

$= \sqrt{3}\sqrt{(2x+5)^2 + 4y^2}$

$= \sqrt{3}\,|2x + 5 + 2yi|$

$= \sqrt{3}\,|2z + 5|.$ ∎

(iii) Let $z_1 = a + bi$, $z_2 = c + di$ for $a, b, c, d \in \mathbb{R}$

$\overline{z_1 z_2} = \overline{(ac - bd) + (ad + bc)i}$

$= (ac - bd) - (ad + bc)i$

$= (a - bi)(c - di)$

$= \bar{z}_1 \bar{z}_2$. $\therefore \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$\overline{z^4} = \overline{z^2\, z^2} = \overline{z^2}\;\overline{z^2} = \overline{z \cdot z} \cdot \overline{z \cdot z} = \bar{z} \cdot \bar{z} \cdot \bar{z} \cdot \bar{z} = \bar{z}^4$ ∎.