Homework #10

1. Consider the integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$

   (i) Use the contour pictured in A) to derive the value $\pi/4$ for the integral.

   (ii) Use the contour pictured in B) to derive the same value for the integral.

2. Use the contour pictured in C) to compute that $\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$

3. (i) Use the contour pictured in A) to compute $\int_0^\infty \frac{x^3 \sin x}{(x^2+1)(x^2+9)} dx$ (ii) Explain why contour B) does not help to compute this integral.

4. (i) Use the integrand $(e^{iaz} - e^{ibz})/z^2$ and the contour pictured in D) to derive

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a) \qquad a \geq 0, b \geq 0$$

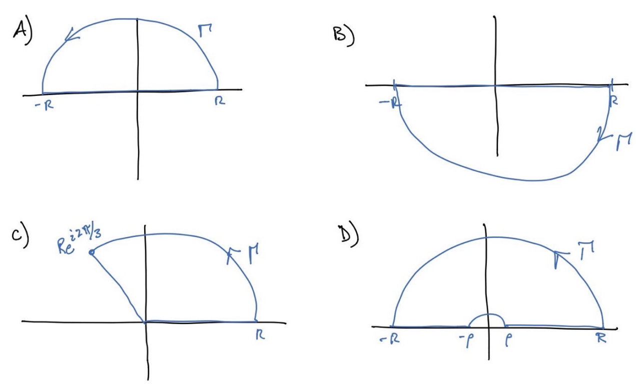
   (ii) using that $1 - \cos(2x) = 2\sin^2 x$, together with the above, show that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

5. Show that

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^\infty \frac{e^{px}}{1+e^x} dx = \frac{\pi}{\sin \pi p}, \qquad p \in \mathbb{C}, 0 < \Re(p) < 1,$$

   by integrating $f(z)$ around the rectangle with vertices R, $R+2\pi i$, $-R+2\pi i$, $-R$, using residues, and letting $R \to \infty$. Make sure to show that the integrals along the vertical sides of the rectangle vanish as $R \to \infty$.



1 (i) The integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$ can be evaluated using the function $f(z)=\frac{1}{(z^2+1)^2}$ and the same simple closed contour as in A . Here

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \int_{C_R} \frac{dz}{(z^2+1)^2} = 2\pi i B,$$

where $B = \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2}$. Since

$$\frac{1}{(z^2+1)^2} = \frac{\phi(z)}{(z-i)^2}, \quad \text{where} \quad \phi(z) = \frac{1}{(z+i)^2},$$

we readily find that $B = \phi'(i) = \frac{1}{4i}$, and so

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2}.$$

If z is a point on C_R, we know from Exercise 1 that

$$|z^2+1| \geq R^2 - 1;$$

thus

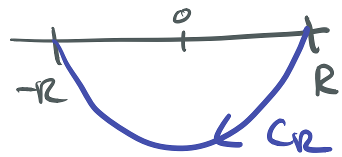
$$\left| \int_{C_R} \frac{dz}{(z^2+1)^2} \right| \leq \frac{\pi R}{(R^2-1)^2} = \frac{\frac{\pi}{R^3}}{\left(1-\frac{1}{R^2}\right)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

The desired result is, then,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}, \quad \text{or} \quad \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

1 (ii) using B we have that

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \int_{C_R} \frac{dz}{(z^2+1)^2} = -2\pi i \operatorname*{Res}_{z=-i}$$

This comes from the curve being clockwise where  . In this case

the residue is at $-i$. we have that

$$\frac{1}{(z^2+i)^2} = \frac{\phi(z)}{(z+i)^2}, \quad \text{where } \phi(z) = \frac{1}{(z-i)^2}$$

$$\operatorname*{Res}_{z=-i} = \phi'(-i) = \frac{-2}{(-2i)^3} = \frac{-2}{-8i^3} = \frac{2}{-8i} = \frac{-1}{4i}$$

This still holds for the curve in B.

$$\left|\int_{C_R} \frac{dz}{(z^2+1)^2}\right| \le \frac{\pi R}{(R^2-1)^2} = \frac{\frac{\pi}{R^3}}{\left(1-\frac{1}{R^2}\right)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

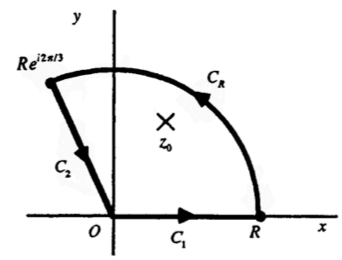
So,

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \lim_{R\to\infty}\left[\int_{-R}^{R} \frac{1}{(x^2+1)^2}\,dx + \int_{C_R} \frac{1}{(z^2+1)^2}\,dz\right]$$

$$= \lim_{R\to\infty} -2\pi i \cdot \left(-\frac{1}{4i}\right) = \frac{\pi}{2}$$

$$\therefore \int_{0}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \boxed{\frac{\pi}{4}}$$

2. The problem here is to establish the integration formula $\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$ using the simple closed contour shown below, where $R > 1$.



There is only one singularity of the function $f(z) = \frac{1}{z^3+1}$, namely $z_0 = e^{i\pi/3}$, that is interior to the closed contour when $R > 1$. According to the residue theorem,

$$\int_{C_1} \frac{dz}{z^3+1} + \int_{C_R} \frac{dz}{z^3+1} + \int_{C_2} \frac{dz}{z^3+1} = 2\pi i \operatorname*{Res}_{z=z_0} \frac{1}{z^3+1},$$

where the legs of the closed contour are as indicated in the figure. Since C_1 has parametric representation $z = r$ $(0 \le r \le R)$,

$$\int_{C_1} \frac{dz}{z^3+1} = \int_0^R \frac{dr}{r^3+1};$$

and, since $-C_2$ can be represented by $z = re^{i2\pi/3}$ $(0 \le r \le R)$,

$$\int_{C_2} \frac{dz}{z^3+1} = -\int_{-C_2} \frac{dz}{z^3+1} = -\int_0^R \frac{e^{i2\pi/3}\,dr}{(re^{i2\pi/3})^3+1} = -e^{i2\pi/3}\int_0^R \frac{dr}{r^3+1}.$$

Furthermore,

$$\operatorname*{Res}_{z=z_0} \frac{1}{z^3+1} = \frac{1}{3z_0^2} = \frac{1}{3e^{i2\pi/3}}.$$

Consequently,

$$(1-e^{i2\pi/3})\int_0^R \frac{dr}{r^3+1} = \frac{2\pi i}{3e^{i2\pi/3}} - \int_{C_R} \frac{dz}{z^3+1}.$$

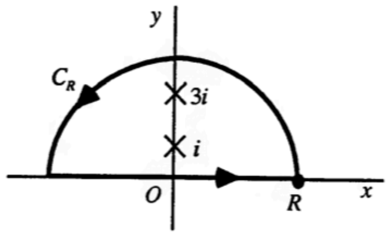
But

$$\left|\int_{C_R} \frac{dz}{z^3+1}\right| \le \frac{1}{R^3-1}\cdot\frac{2\pi R}{3} \to 0 \text{ as } R \to \infty.$$

This gives us the desired result, with the variable of integration r instead of x:

$$\int_0^R \frac{dr}{r^3+1} = \frac{2\pi i}{3(e^{i2\pi/3} - e^{i4\pi/3}\cdot e^{-i6\pi/3})} = \frac{2\pi i}{3(e^{i2\pi/3} - e^{-i2\pi/3})} = \frac{\pi}{3\sin(2\pi/3)} = \frac{2\pi}{3\sqrt{3}}.$$

3.i) In order to evaluate the integral $\int_0^{\infty} \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)}$, we introduce here the function $f(z) = \frac{z^3}{(z^2+1)(z^2+9)}$. Its singularities in the upper half plane are i and $3i$, and we consider the simple closed contour shown below, where $R > 3$.



Since

$$\operatorname*{Res}_{z=i}\left[f(z)e^{iz}\right] = \frac{z^3 e^{iz}}{(z+i)(z^2+9)}\Bigg]_{z=i} = -\frac{1}{16e}$$

and

$$\operatorname*{Res}_{z=3i}\left[f(z)e^{iz}\right] = \frac{z^3 e^{iz}}{(z^2+1)(z+3i)}\Bigg]_{z=3i} = \frac{9}{16e^3},$$

the residue theorem tells us that

$$\int_{-R}^{R} \frac{x^3 e^{ix}\, dx}{(x^2+1)(x^2+9)} + \int_{C_R} f(z)e^{iz} dx = 2\pi i\left(-\frac{1}{16e} + \frac{9}{16e^3}\right),$$

or

$$\int_{-R}^{R} \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8e}\left(\frac{9}{e^2} - 1\right) - \operatorname{Im}\int_{C_R} f(z)e^{iz} dz.$$

Now if z is a point on C_R, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R}{(R^2-1)(R^2-9)} \quad \text{as } R \to \infty.$$

So, in view of Jordan's Lemma

$$\left|\operatorname{Im}\int_{C_R} f(z)e^{iz} dz\right| \le \left|\int_{C_R} f(z)e^{iz} dz\right| \to 0 \text{ as } R \to \infty;$$

and this means that

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8e}\left(\frac{9}{e^2} - 1\right), \quad \text{or} \quad \int_{0}^{\infty} \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{16e}\left(\frac{9}{e^2} - 1\right).$$

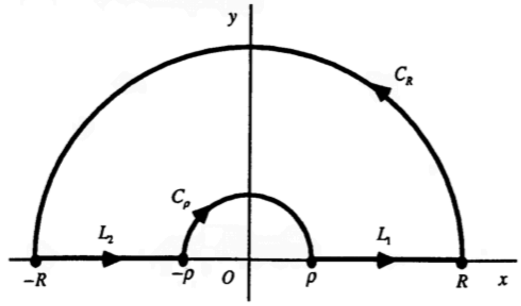
(ii) Jordan's lemma only holds in the upper half plane. This is because in the integral we get a term that looks like $e^{-R\sin\theta}$ in the upper half plane and in the lower plane it would be $e^{R\sin\theta}$

This term grows as $R \to \infty$ instead of shrinking like $e^{-R\sin\theta}$.

4.i) The main problem here is to derive the integration formula

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0, b \ge 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z) = \frac{e^{iaz}-e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 0,$$

or

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since L_1 and $-L_2$ have parametric representations

$$L_1 : z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r \ (\rho \le r \le R),$$

we can see that

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_\rho^R \frac{e^{iar}-e^{ibr}}{r^2}\,dr + \int_\rho^R \frac{e^{-iar}-e^{-ibr}}{r^2}\,dr$$

$$= \int_\rho^R \frac{(e^{iar}+e^{-iar})-(e^{ibr}+e^{-ibr})}{r^2}\,dr = 2\int_\rho^R \frac{\cos(ar)-\cos(br)}{r^2}\,dr.$$

Thus

$$2\int_\rho^R \frac{\cos(ar)-\cos(br)}{r^2}\,dr = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

In order to find the limit of the first integral on the right here as $\rho \to 0$, we write

$$f(z) = \frac{1}{z^2}\left[\left(1+\frac{iaz}{1!}+\frac{(iaz)^2}{2!}+\frac{(iaz)^3}{3!}+\cdots\right)-\left(1+\frac{ibz}{1!}+\frac{(ibz)^2}{2!}+\frac{(ibz)^3}{3!}+\cdots\right)\right]$$

$$= \frac{i(a-b)}{z}+\cdots \quad (0<|z|<\infty).$$

From this we see that $z=0$ is a simple pole of $f(z)$, with residue $B_0 = i(a-b)$. Thus

$$\boxed{\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = -B_0\pi i = -i(a-b)\pi i = \pi(a-b).}$$

As for the limit of the value of the second integral as $R \to \infty$, we note that if z is a point on C_R, then

$$f(z) \le \frac{|e^{iaz}|+|e^{ibz}|}{|z|^2} = \frac{e^{-ay}+e^{-by}}{R^2} \le \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0 \ \text{ as } \ R \to \infty.$$

It is now clear that letting $\rho \to 0$ and $R \to \infty$ yields

$$2\int_0^\infty \frac{\cos(ar)-\cos(br)}{r^2}\,dr = \pi(b-a).$$

4.ii. This is the desired integration formula, with the variable of integration r instead of x. Observe that when $a=0$ and $b=2$, that result becomes

$$\int_0^\infty \frac{1-\cos(2x)}{x^2}\,dx = \pi.$$

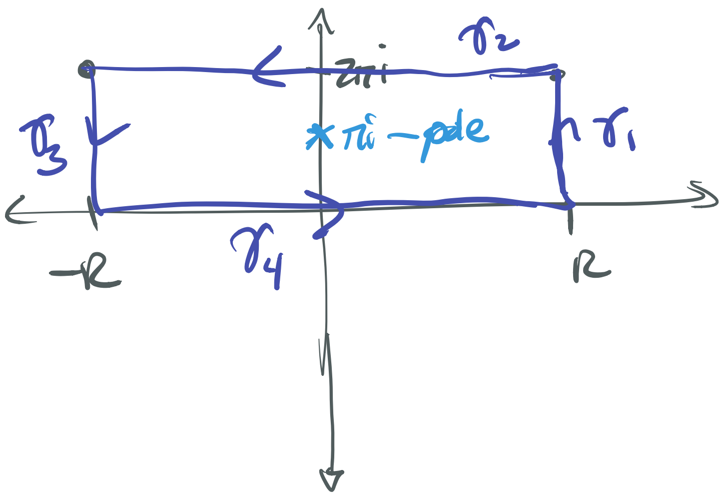
But $\cos(2x) = 1-2\sin^2 x$, and we arrive at

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

5. Show that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\,dx = \frac{\pi}{\sin \pi p}, \qquad p \in \mathbb{C}, 0 < \Re(p) < 1,$$

by integrating $f(z)$ around the rectangle with vertices R, $R+2\pi i$, $-R+2\pi i$, $-R$, using residues, and letting $R \to \infty$. Make sure to show that the integrals along the vertical sides of the rectangle vanish as $R \to \infty$.



$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\pi i} f(z).$$

$$\operatorname*{Res}_{z=\pi i} f(z) = \lim_{z\to\pi i} (z-\pi i)\frac{e^{pz}}{1+e^z} = \lim_{z\to\pi i} \frac{p(z-\pi i)e^{pz} + e^{pz}}{e^z}$$

$$= -e^{\pi p i}.$$

$$\left|\int_{\gamma_1} f(z)\,dz\right| = \left|\int_0^{2\pi} \frac{e^{p(R+xi)}}{1+e^{(R+xi)}}\,dx\right|$$

$$= \left| \int_0^{2\pi} \frac{e^{pR} e^{pxi}}{1 + e^{R} e^{xi}} dx \right|$$

$$= \left| \int_0^{2\pi} e^{(p-1)R} \left[\frac{e^{pxi}}{e^{-R} + e^{xi}} \right] dx \right|$$

$$= e^{(p-1)R} \left| \int_0^{2\pi} e^{pxi} / e^{-R} + e^{xi} \, dx \right|$$

Note that if R is sufficiently large

Then $|e^{-R} + e^{xi}| \geq 1/2$

$$\leq e^{(p-1)R} \int_0^{2\pi} 2 \, |e^{pxi}| \, dx$$

$$\leq e^{(p-1)R} \, 4\pi \to 0 \quad \text{as } R \to \infty$$

as long as $0 < p < 1$.

$$\left| \int_{\gamma_3} f(z) dz \right| = \left| \int_{2\pi}^{0} \frac{e^{-pR} e^{xi}}{1 + e^{-R} e^{xi}} dx \right|$$

$$\leq \left| \int_{2\pi}^{0} \frac{e^{-pR}}{1 - e^{-R}} \right| dx = \frac{2\pi e^{-pR}}{1 - e^{-R}} \to 0$$

as $R \to \infty$

$$\int_{\gamma_2} f(z)\,dz = \int_{R}^{-R} \frac{e^{px}\, e^{p2\pi i}}{1+e^{x}}\,dx = -e^{2\pi i p}\int_{-R}^{R}\frac{e^{px}}{1+e^{x}}\,dx$$

$$\int_{\gamma_4} f(z)\,dz = \int_{R}^{-R} \frac{e^{px}}{1+e^{x}}\,dx.$$

so

$$\left[1-e^{2\pi i p}\right]\int_{-R}^{R}\frac{e^{px}}{1+e^{x}}\,dx + \int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$

$$= 2\pi i\left[-e^{p\pi i}\right]$$

$$\Rightarrow \int_{-R}^{R}\frac{e^{px}}{1+e^{x}}\,dx + \left[1-e^{2\pi i p}\right]^{-1}\left[\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz\right]$$

$\neq 0$ since $p \neq 0, 1$

$$= \frac{-2\pi i\, e^{p\pi i}}{1-e^{2\pi i p}} = \frac{-2\pi i}{e^{-p\pi i}-e^{p\pi i}}$$

$$= \frac{\pi}{\sin(\pi p)}$$

so

$$\int_{-\infty}^{\infty}\frac{e^{px}}{1+e^{x}}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{px}}{1+e^{x}}\,dx$$

$$= \lim_{R \to \infty} \left[1 - e^{2\pi i p}\right]^{-1} \left[\overbrace{\int_{\Gamma_1} f(z)\,dz}^{\to 0} + \overbrace{\int_{\Gamma_2} f(z)\,dz}^{\to 0} \right] + \frac{\pi}{\sin(\pi p)}$$

$$= \frac{\pi}{\sin(\pi p)}.$$