Homework #11

1. Show that the mapping $w = iz$ is a rotation by $\pi/2$. Then find the image of the strip $0 < \Re[z] < 1$.

2. Give a geometric interpretation of the mapping $w = iz + i$. Then find the image of the half plane $\Re[z] > 0$.

3. Show that when a circle is transformed into a circle under the transformation $w = 1/z$, the center of the original circle is *never* mapped onto the center of the image circle.

4. (i) Show that the transformation $w = 1/z$ maps the hyperbola $x^2 - y^2 = 1$ in the z-plane into the lemniscate $\rho^2 = \cos 2\phi$ in the $w = \rho e^{i\phi}$-plane. In particular, what branch of the hyperbola gets mapped onto what lobe of the lemniscate. Sketch both. In what direction is the particular lobe of the lemniscate traversed if the corresponding branch of the hyperbola is traversed from negative to positive values of y? Indicate this in the sketch.

   HINT: Use the fact that

   $$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}.$$

   (ii) Onto what object does $1/z$ map the hyperbola $x^2 - y^2 = -1$? Sketch the hyperbola and its image. As in (i), discuss the traversing directions, and include the results in the sketch.

5. Derive that the general linear fractional transformation that maps

   (i) 0 to 0 and 1 to 1 is

   $$w = \frac{z}{(1 - \lambda)z + \lambda},$$

   where λ is an arbitrary complex parameter;

   (ii) 0 to 1 and 1 to 0 is

   $$w = \frac{z - 1}{\lambda z - 1} = \mu \frac{z - 1}{z - \mu},$$

   where $\lambda = 1/\mu$ is an arbitrary complex parameter;

   To what point is ∞ mapped and what point is mapped to ∞ in each case?

6. (i) If we interpret a linear fractional transformation

   $$T(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0,$$

   as a transformation of the z-plane onto itself, we can ask if T has any *fixed points*, i.e., points z that satisfy the equation $T(z) = z$. Show that, except for the trivial cases ($c = 0$ and $a = d$, or $b = c = 0$), there exist two fixed points. If $(d - a)^2 + 4bc = 0$, show that these two points coincide.

   (ii) If a linear fractional transformation $T(z)$ has two fixed points, say α and β, use the cross-ratio to show that $T(z)$ is equivalent to the equation

   $$\frac{w - \alpha}{w - \beta} = \lambda \frac{z - \alpha}{z - \beta},$$

   where $\lambda \neq 0$ is a complex constant.

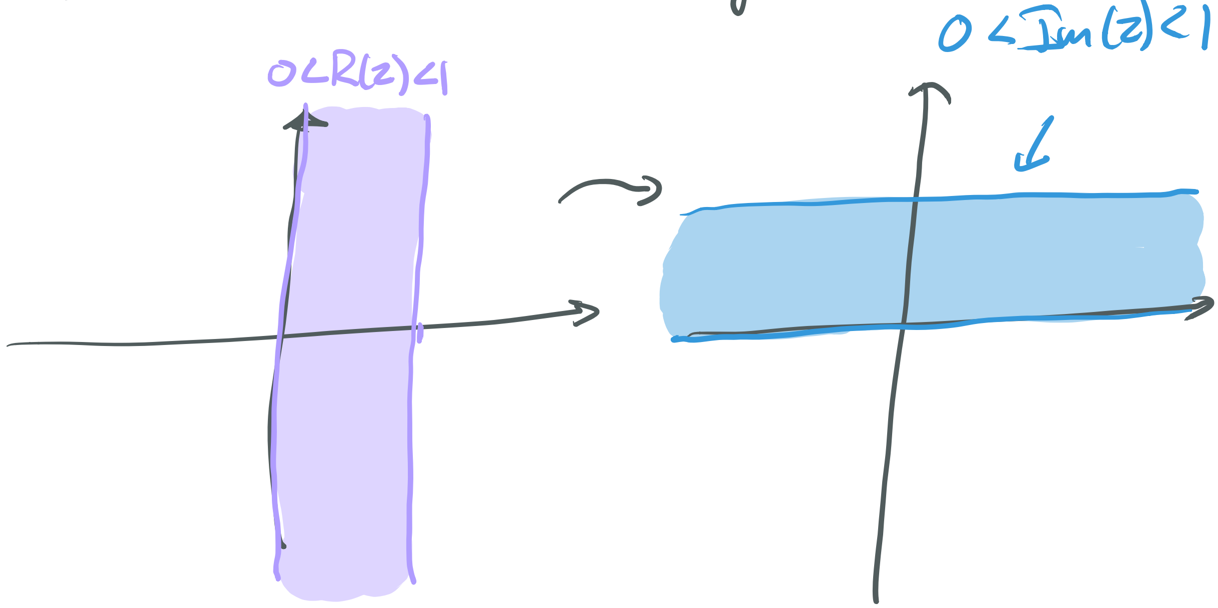
   (iii) Use (ii) to reproduce the result of 5 (i).

1. Show that the mapping $w = iz$ is a rotation by $\pi/2$. Then find the image of the strip $0 < \Re[z] < 1$.

We can rewrite $z = re^{i\theta}$ and $i = e^{\frac{\pi}{2}i}$

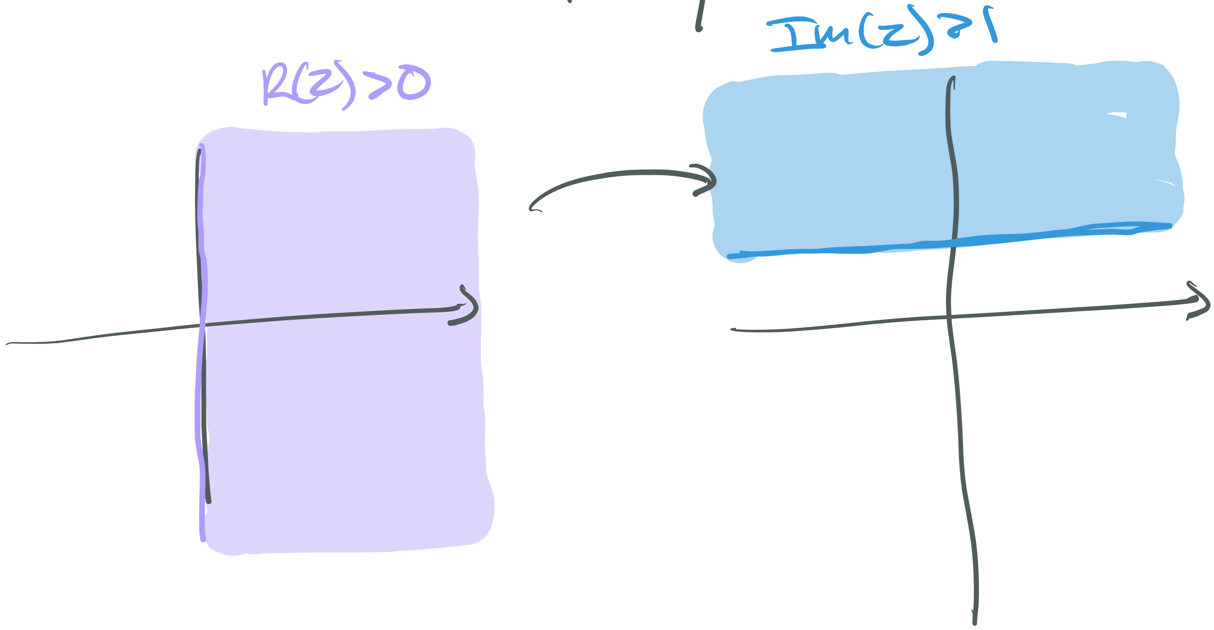
so $w = e^{\frac{\pi}{2}i} re^{i\theta} = r\, e^{i(\theta + \frac{\pi}{2})}$. so

$w = \pi/2$ is a rotation by $\pi/2$.

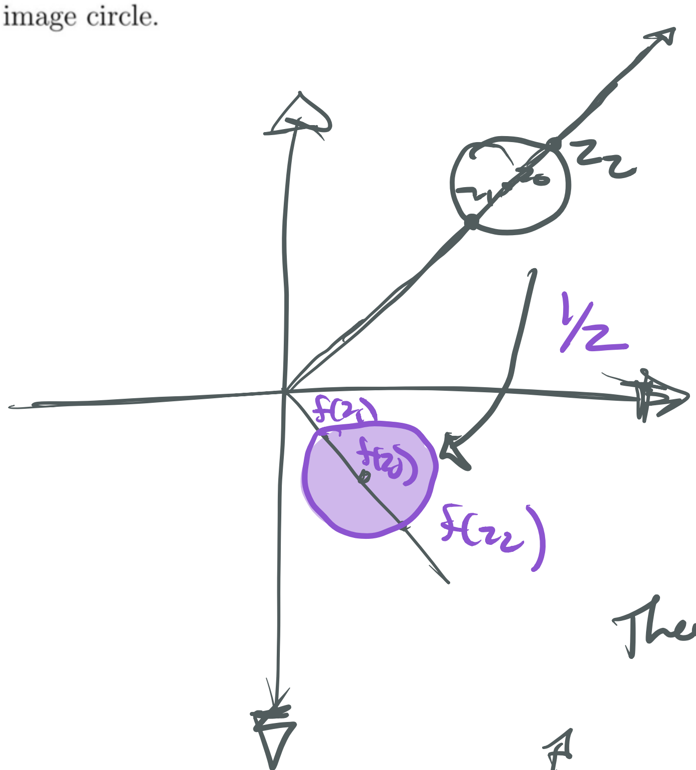


2. Give a geometric interpretation of the mapping $w = iz + i$. Then find the image of the half plane $\Re[z] > 0$.

$w = iz + i$ is a rotation by $\frac{\pi}{2}$ and then a shift up by 1.



3. Show that when a circle is transformed into a circle under the transformation $w = 1/z$, the center of the original circle is *never* mapped onto the center of the image circle.



Note: z_0 cannot be 0.

Then $f(z_0) = \dfrac{1}{r_1} e^{i(-\theta)}$

$f(z_1) = \dfrac{1}{r_1 - R} e^{i(-\theta)}$

$f(z_2) = \dfrac{1}{r_1 + R} e^{i-\theta}$

$z_0 = r_1 e^{i\theta}$

$z_1 = (r_1 - R) e^{i\theta}$

$z_2 = (r_1 + R) e^{i\theta}$

If $f(z_0)$ is the center then we need $|f(z_0) - f(z_1)| = |f(z_2) - f(z_0)|$

since z_1, z_2 will still be on the boundary.

$$|f(z_1)-f(z_0)| = \left|\frac{1}{r_1} - \frac{1}{r_1-R}\right| = \frac{1}{r_1-R} - \frac{1}{r_1}$$

$$|f(z_2)-f(z_0)| = \left|\frac{1}{r_1} - \frac{1}{r_1+R}\right| = \frac{1}{r_1} - \frac{1}{r_1+R}$$

$$\Rightarrow \frac{1}{r_1-R} + \frac{1}{r_1+R} = \frac{2}{r_1}$$

$$\frac{r_1}{r_1-R} + \frac{r_1}{r_1+R} = 2$$

$$\frac{r_1^2 + r_1R + r_1^2 - r_1R}{r_1^2 - R^2} = 2$$

$$\frac{r_1^3}{r_1^2-R^2} = 1$$

only happens if $R=0$

4. (i) Show that the transformation $w = 1/z$ maps the hyperbola $x^2 - y^2 = 1$ in the z-plane into the lemniscate $\rho^2 = \cos 2\phi$ in the $w = \rho e^{i\phi}$-plane. In particular, what branch of the hyperbola gets mapped onto what lobe of the lemniscate. Sketch both. In what direction is the particular lobe of the lemniscate traversed if the corresponding branch of the hyperbola is traversed from negative to positive values of y? Indicate this in the sketch.

HINT: Use the fact that

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}.$$

(ii) Onto what object does $1/z$ map the hyperbola $x^2 - y^2 = -1$? Sketch the hyperbola and its image. As in (i), discuss the traversing directions, and include the results in the sketch.

(i) $1 = x^2 - y^2 = \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2$

$$1 = \frac{1}{4}\left(z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2\right)$$

$$1 = \frac{1}{2}\left(z^2 + \bar{z}^2\right)$$

$$2 = z^2 + \bar{z}^2$$

$$2 = \rho^2 e^{i2\phi} + r^2 e^{-i2\phi}$$

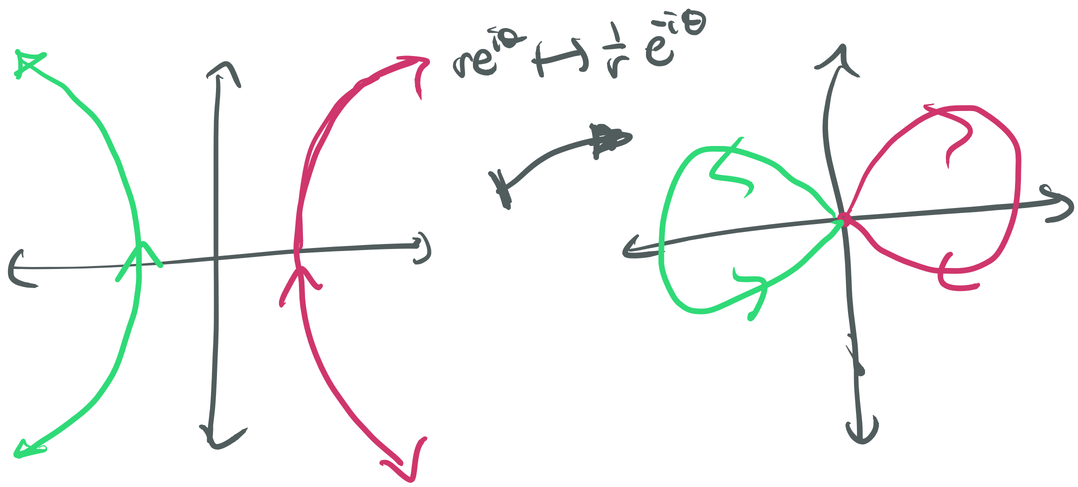
Now $\rho^2 e^{i2\theta} \mapsto \frac{1}{\rho^2} e^{-i2\phi}$ $\qquad \rho^2 e^{-i2\phi} \mapsto \frac{1}{\rho^2} e^{i2\phi}$

$\Rightarrow$

$$2 = \frac{1}{\rho^2} e^{-i2\phi} + \frac{1}{\rho^2} e^{i2\phi}$$

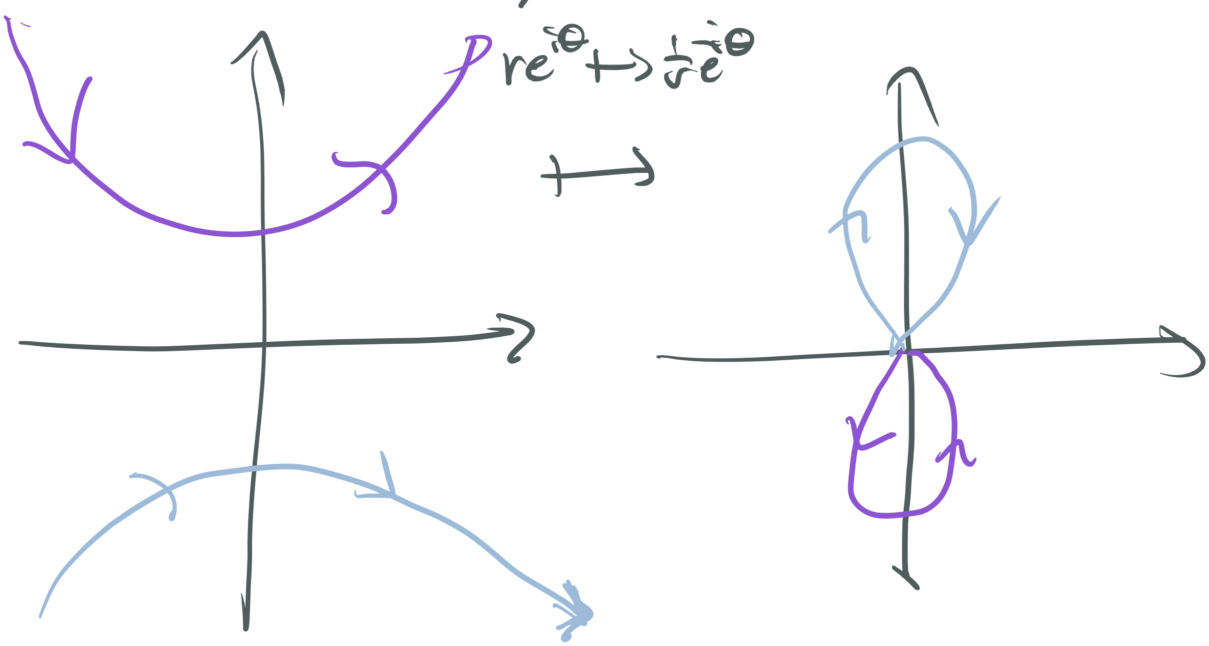
$\Rightarrow$ $2\rho^2 = e^{-i2\phi} + e^{i2\phi} = 2\cos(2\phi)$

$\Rightarrow$ $\rho^2 = \cos(2\phi).$



(ii) Notice that we just have a negative sign on the LHS. So we get

$$-\rho^2 = \cos(2\phi)$$



5. Derive that the general linear fractional transformation that maps

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where λ is an arbitrary complex parameter;

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$$w = \frac{z-1}{\lambda z-1} = \mu\frac{z-1}{z-\mu},$$

where $\lambda = 1/\mu$ is an arbitrary complex parameter;

To what point is ∞ mapped and what point is mapped to ∞ in each case?

(i) $F(z) = \frac{az+b}{cz+d}$ so

$F(0) = \frac{b}{d} = 0 \Rightarrow b = 0$

$F(1) = \frac{a}{c+d} = 1 \Rightarrow a = c+d$

or equivalently $1 = \frac{c}{a} + \frac{d}{a}$

Take $a=1$ the $1 = c+d$

$d=\lambda$ the $c = 1-\lambda$

Hence, $F(z) = \frac{1\cdot z+0}{(1-\lambda)z+\lambda} = \frac{z}{(1-\lambda)z+\lambda}$

$F(\infty) = \frac{1}{1-\lambda}$ and $F\left(\frac{-\lambda}{1-\lambda}\right) = \infty$

(ii) $F(z) = \dfrac{az+b}{cz+d}$

$F(0) = \dfrac{b}{d} = 1$, $F(1) = \dfrac{a+b}{c+d} = 0$

$\Rightarrow$ $b = d$ and $a = -b$

So, $a = -b = -d$, variable c is free.

Take $a=1$ and $c=\lambda$ then,

$$F(z) = \frac{z-1}{\lambda z-1} = \frac{1}{\lambda}\,\frac{z-1}{z-1/\lambda} = \mu\left(\frac{z-1}{z-\mu}\right).$$

$F(\infty) = \frac{1}{\lambda} = \mu$ and $F\left(\frac{1}{\lambda}\right) = F(\mu) = \infty$.

6. (i) If we interpret a linear fractional transformation

$$T(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0,$$

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where $\lambda \neq 0$ is a complex constant.

(iii) Use (ii) to reproduce the result of 5 (i).

(i) $$\frac{az+b}{cz+d} = z$$

Then, $$az+b = cz^2+dz$$

$$0 = cz^2 + (d-a)z - b$$

So, z solves a quadratic so it has two solutions as long as $c \neq 0$, with possibly a double solution. By the quadratic formula this happens when $(d-a)^2 - 4(c(-b)) = 0$ which simplifies to $(d-a)^2 + 4bc = 0$.

(ii) Let $z_1 = w_1 = \alpha$ and $z_3 = w_3 = \beta$.

$$\frac{(w-\alpha)}{(w-\beta)} \frac{(w_2-\beta)}{(w_2-\alpha)} = \frac{(z-\alpha)}{(z-\beta)} \frac{(z_2-\beta)}{(z_2-\alpha)}$$

$$\Rightarrow \quad \frac{w-\alpha}{w-\beta} = \underbrace{\left(\frac{w_2-\alpha}{w_2-\beta}\right)\left(\frac{z_2-\beta}{z_2-\alpha}\right)}_{\lambda} \frac{z-\alpha}{z-\beta}$$

$$\frac{w-\alpha}{w-\beta} = \lambda \frac{z-\alpha}{z-\beta}$$

(iii) $\alpha = 1 \quad \beta = 0$

$$\frac{(w-1)(w_2-0)}{w(w_2-1)} = \frac{(z-1)(z_2)}{z(z_2-1}$$

$$\frac{w-1}{w} = \underbrace{\frac{w_2-1}{w_2} \quad \frac{z_2}{z_2-1}}_{\lambda} \cdot \frac{z-1}{z}$$

$$(w-1)z = \lambda(z-1)w$$

$$w(z-(z-1)\lambda) = z$$

$$w = \frac{z}{z(1-\lambda)+\lambda}$$