Homework #12

1. In class we discussed a mapping that mapped the upper-half plane to inside the unit circle. Find a mapping that maps the lower-half plane to inside the unit circle.

2. Determine the angle of rotation at the point $z = 2 + i$ when the transformation is $w = z^2$ and illustrate it for some particular curve. Determine the scale factor of the transformation.

3. In what domain is $w = \sin z$ a conformal mapping? In what domain is $w = \cosh z$ a conformal mapping?

4. If $u = x^3 - 3xy^2$, find a function $v(x,y)$ such that $f(x,y) = u(x,y) + i v(x,y)$ for $z = x + iy$ is analytic. On what domain does $f(z)$ define a conformal map?

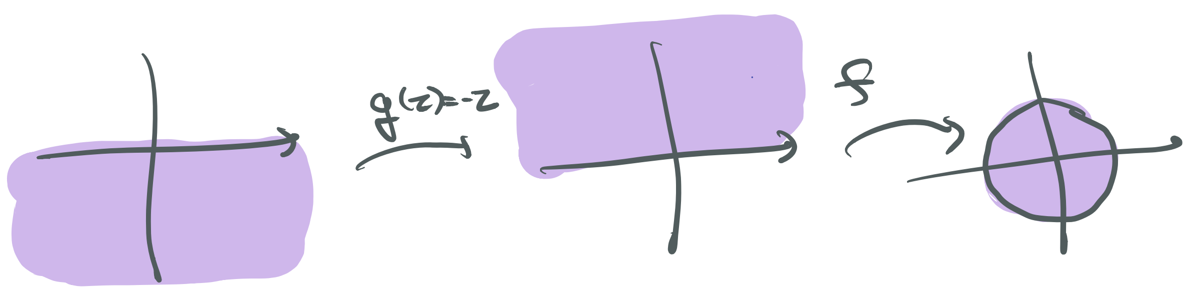
1. In class we discussed a mapping that mapped the upper-half plane to inside the unit circle. Find a mapping that maps the lower-half plane to inside the unit circle.

The map $f(z) = \dfrac{z-i}{z+i}$ maps the upper half plane to the unit disk.

So if we have a map from the lower half plane to the upper half plane we can compose the two maps.

$g(z) = -z$ sends the lower half plane to the upper. Hence,

$$(f \circ g)(z) = \frac{-z-i}{-z+i}$$ sends the lower half plane to the unit disk



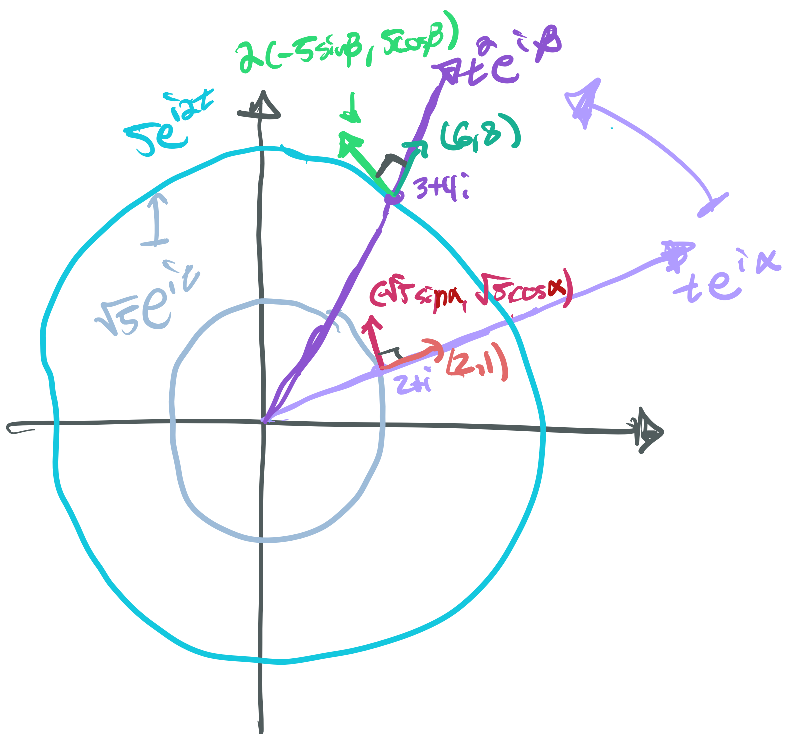
2. Determine the angle of rotation at the point $z = 2 + i$ when the transformation is $w = z^2$ and illustrate it for some particular curve. Determine the scale factor of the transformation.

$$z \longmapsto z^2 \qquad f'(z) = 2z$$

$$2+i \longmapsto 3+4i \qquad f'(2+i) = 4+2i$$

$$\text{Angle of Rotation} = \operatorname{Arctan}\left(\tfrac{2}{4}\right) = \operatorname{Arctan}\left(\tfrac{1}{2}\right)$$

$$\text{Scale factor} = |f'(2+i)| = |4+2i| = \sqrt{20}$$



3. In what domain is $w = \sin z$ a conformal mapping? In what domain is $w = \cosh z$ a conformal mapping?

(i) $f(z) = \sin(z)$ is conformal as long as $f'(z) = \cos(z) \neq 0$. So we need,

$z \neq k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$.

(ii) $f(z) = \cosh(z)$ is conformal as long as $f'(z) = \sinh(z) = \frac{e^z - e^{-z}}{2} \neq 0$

Recall from previous homeworks this is when $z \neq k\pi i$

4. If $u = x^3 - 3xy^2$, find a function $v(x,y)$ such that $f(x,y) = u(x,y) + i\,v(x,y)$ for $z = x+iy$ is analytic. On what domain does $f(z)$ define a conformal map?

By Cauchy Riemann Equations we need

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{and}$$

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = -6xy$$

Hence,

$$v = 3x^2y - y^3 + g(x)$$

$$v = 3x^2y + h(y)$$

So, $h(y) = -y^3$ and $g(x) = 0$

$$\boxed{v(x,y) = 3x^2y - y^3}$$

$$f'(z) = u_x + i v_x = (3x^2 - 3y^2) + i\,(6xy)$$

$$f'(z) \neq 0 \text{ if } 3x^2 - 3y^2 \neq 0 \text{ or } 6xy \neq 0$$

$x \neq \pm y$ $\quad$ x and $y \neq 0$

So, $f(z)$ is conformal on $\mathbb{C} \setminus \{0\}$.