Homework #2

1. Find the principal argument of z

   (i) $z = -2/(1 + \sqrt{3}i)$

   (ii) $z = (\sqrt{3} - i)^6$

2. Compute the following directly, and also by first converting individual factors to exponential form

   (i) $i(1 - \sqrt{3}i)(\sqrt{3} + i)$

   (ii) $5i/(2 + i)$

3. Find all the solutions to $z^8 = 32$

4. This problem is to derive Euler's Identity with differential equations

   (i) Show $\sin t$ and $\cos t$ are both solutions to $y'' + y = 0$

   (ii) Solve the initial value problem $y(0) = 1, y'(0) = 0$.

   (iii) Show e^{it} and e^{-it} solve the differential equation, and find the solution to the above initial value problem, thus finding an expression for $\cos t$.

   (iv) Repeat with the initial value problem $y(0) = 0, y'(0) = 1$, thus finding an expression for $\sin t$

   (v) Using the results from (iii) and (iv) solve for e^{it}, arriving at Euler's Identity.

5. Show that
$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}$$
   if z lies on the circle centered at the origin with radius 2. Hint: try factoring the denominator.

6. Write the function $f(z) = 1/z$ in the form $f = u(x, y) + iv(x, y)$ for $z = x + iy$. The region $|z| < 1$ is mapped to what region by this function?

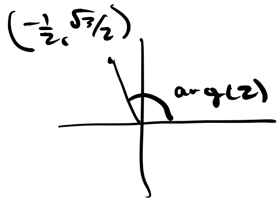
7. Write the function $f(z) = z^3$ in the form $f = R(r, \theta)e^{i\Theta(r,\theta)}$ for $z = re^{i\theta}$. The first quadrant is mapped to what region by this function?

1. Find the principal argument of z

(i) $z = -2/(1+\sqrt{3}i)$

(ii) $z = (\sqrt{3}-i)^6$

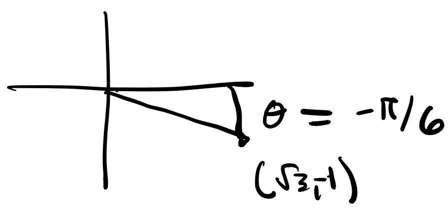
(i) $z = \dfrac{-2(1-\sqrt{3}i)}{4} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$



$\arg(z) = \arctan((\sqrt{3}/2)/(1/2)) = \boxed{\frac{2\pi}{3}}$

(ii) $\sqrt{3} - i = 2e^{-i\pi/6} \Rightarrow (\sqrt{3}-i)^6 = 2^6 e^{(-i\pi/6)6}$

$= 64e^{-i\pi}$



so $\arg(z) = \boxed{\pi}$

Remember that the principal argument is in $(-\pi, \pi]$

2. Compute the following directly, and also by first converting individual factors to exponential form

(i) $i(1-\sqrt{3}i)(\sqrt{3}+i)$

(ii) $5i/(2+i)$

(i) $$i(1-\sqrt{3}i)(\sqrt{3}+i) = e^{i\pi/2}\left(2e^{-i\pi/3}\right)\left(2e^{i\pi/6}\right)$$
$$= 4e^{i\pi/3}$$
$$= \boxed{2+2\sqrt{3}i}$$

$$i(1-\sqrt{3}i)(\sqrt{3}+i) = i(\sqrt{3}-3i+i+\sqrt{3})$$
$$= i(2\sqrt{3}-2i)$$
$$= \boxed{2+2\sqrt{3}i}$$

(ii) $$\frac{5i}{2+i} = \frac{5i(2-i)}{5} = \boxed{2i+1}$$

$$\frac{5i}{2+i} = \left(5e^{i\pi/2}\right)\left(\frac{1}{\sqrt{5}}e^{-\arctan(1/2)}\right)$$
$$= \sqrt{5}\, e^{i(\pi/2-\arctan(1/2))}$$
$$= \sqrt{5}\, e^{i\arctan(2)}$$
$$= \sqrt{5}\left(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i\right) = \boxed{1+2i}$$

3. Find all the solutions to $z^8 = 32$

$$z = r e^{i\theta}$$

$$z^8 = r^8 e^{i8\theta} = 32$$

Want $8\theta = 2\pi k$ for $k \in \mathbb{Z}$
and $0 \leq \theta < 2\pi$

$$\theta = 0, \frac{2\pi}{8}, \frac{4\pi}{8}, \frac{6\pi}{8}, \frac{8\pi}{8}, \frac{10\pi}{8}, \frac{12\pi}{8}, \frac{14\pi}{8}$$

$$r = (32)^{1/8} = 2^{5/8}$$

So the solutions are

$$\boxed{\begin{aligned} z = {} & 2^{5/8}, \quad 2^{5/8}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) \\ & 2^{5/8}i, \quad 2^{5/8}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) \\ & -2^{5/8}, \quad 2^{5/8}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \\ & -2^{5/8}i, \quad 2^{5/8}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \end{aligned}}$$

4. This problem is to derive Euler's Identity with differential equations

(i) Show $\sin t$ and $\cos t$ are both solutions to $y'' + y = 0$

(ii) Solve the initial value problem $y(0) = 1, y'(0) = 0$.

(iii) Show e^{it} and e^{-it} solve the differential equation, and find the solution to the above initial value problem, thus finding an expression for $\cos t$.

(iv) Repeat with the initial value problem $y(0) = 0, y'(0) = 1$, thus finding an expression for $\sin t$

(v) Using the results from (iii) and (iv) solve for e^{it}, arriving at Euler's Identity.

(i) $(\sin(t))'' + \sin(t) = -\sin(t) + \sin(t) = 0$ ✓

$(\cos(t))'' + \cos(t) = -\cos(t) + \cos(t) = 0$ ✓

(ii) $y = A\cos t + B\sin t$

so, $1 = y(0) = A\cos(0) + B\sin(0) = A$

$0 = y'(0) = -A\sin(0) + B\cos(0) = B$

$\Rightarrow y = \cos t$

(iii) $(e^{it})'' + e^{it} = (i)^2 e^{it} + e^{it} = 0$

$(e^{-it})'' + e^{-it} = (-i)^2 e^{-it} + e^{-it} = 0$

$y = Ae^{it} + Be^{-it}$

$1 = y(0) = A + B$

$0 = y'(0) = iA - iB$

$\Rightarrow$ $A + B = 1$, $A - B = 0$

$\Rightarrow A = B = 1/2.$

$$\Rightarrow \boxed{\cos(t) = \frac{e^{it} + e^{-it}}{2}}$$

$y = A\cos t + B\sin t$

(iv) $0 = y(0) = A\cos(0) + B\sin(0) = A$

$1 = y'(0) = -A\sin(0) + B\cos(0) = B$

$\Rightarrow \quad y = \sin(t).$

$y = Ae^{it} + Be^{-it}$

$0 = y(0) = A + B \qquad \Rightarrow \qquad A + B = 0$

$1 = y'(0) = iA - iB \qquad -A + B = i$

$A = -i/2 \quad B = i/2$

$$\Rightarrow \quad \sin t = \frac{ie^{-it} - ie^{it}}{2}$$

Using the equations for $\sin(t)$ and $\cos(t)$ we get

$$\cos t + i\sin t = \frac{e^{it} + e^{-it}}{2} + \frac{e^{it} - e^{-it}}{2}$$

$$= e^{it}.$$

∎

Remark: In this proof we use the uniqueness of solutions to differential equations.

5. Show that

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}$$

if z lies on the circle centered at the origin with radius 2. Hint: try factoring the denominator.

$$\left|\frac{1}{z^4-4z^2+3}\right| = \left|\frac{1}{(z^2-1)(z^2-3)}\right| = \frac{1}{|z^2-1|}\,\frac{1}{|z^2-3|}$$

$$\leq \frac{1}{|z^2|-1}\,\frac{1}{|z^2|-3}$$

$$= \frac{1}{4-1}\,\frac{1}{4-3}$$

$$= \frac{1}{3} \quad \blacksquare.$$

Remark: $|a-b| \geq |a|-|b|$

and if $|a| \geq |b|$

then $\frac{1}{|a|} \leq \frac{1}{|b|}$

6. Write the function $f(z) = 1/z$ in the form $f = u(x, y) + iv(x, y)$ for $z = x + iy$. The region $|z| < 1$ is mapped to what region by this function?

$$f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$$

$$u(x, y) = \frac{x}{x^2+y^2}, \quad v(x, y) = \frac{-y}{x^2+y^2}.$$

$|z|<1$ is mapped to $|z|>1$.

7. Write the function $f(z) = z^3$ in the form $f = R(r,\theta)e^{i\Theta(r,\theta)}$ for $z = re^{i\theta}$. The first quadrant is mapped to what region by this function?

$$f(z) = (re^{i\theta})^3 = r^3 e^{i3\theta}$$

$$R(r,\theta) = r^3 \qquad \Theta(r,\theta) = 3\theta$$

If z is in the first quadrant then

$$z = re^{i\theta} \quad \text{for} \quad r>0 \quad 0<\theta<\pi/2$$

so $f(z) = r^3 e^{i3\theta} = ae^{ib}$ for $3\cdot 0 < b < 3\cdot\pi/2,\ a>0$

since $a>0$ and $0<b<3\pi/2$ we get that $f(z)$ maps the first quadrant to the first 3 quadrants.

Remark: In problems 6 and 7 the functions are one-to-one on the restricted sets.