Homework #5

1. Evaluate the following integrals:

   (i) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

   (ii) $\int_0^{\pi/6} e^{i2t} dt$

   (iii) $\int_0^{\infty} e^{-zt} dt$ for $z \in \mathbb{C}$. For what values of z does the integral converge?

2. Recall the mean-value theorem from real-valued calculus: If $f(t) : \mathbb{R} \to \mathbb{R}$ is continuous function on the interval $a \leq t \leq b$ then there exists some $c \in [a, b]$ such that

$$\int_a^b f(t)dt = f(c)(b - a).$$

   (i) Does it remain true for $f(t) : \mathbb{R} \to \mathbb{C}$ (yes or no, no proof needed)?

   (ii) Try the mean-value theorem out on $f(t) = t + it^2$ for $t \in [0, 1]$ and $f(t) = e^{2it}$ for $t \in [0, \pi]$.

   (iii) Explain why the mean-value theorem failed in the above two cases.

   (iv) Come up with an (non-constant function) example for which the mean-value theorem holds.

3. Evaluate $\int_\Gamma \bar{z} dz$ from $z = 0$ to $z = 4 + 2i$ along Γ given by

   (i) $z = t^2 + it$

   (ii) The line from $z = 0$ to $z = 2i$ then along the line $z = 2i$ to $z = 4 + 2i$.

   (iii) Why do the above two results not violate the theorem that contour integrals are path independent?

4. Integrate $\int_\Gamma z^i dz$ for the principal branch of

$$z^i = e^{i\text{Log}z} \qquad -\pi < \text{Arg}z \leq \pi$$

   and Γ the semicircle of unit radius in the upper half plane.

5. Take Γ to be the upper half of a circle of radius R. Find M in terms of R such that

$$\left| \int_\Gamma \frac{dz}{2 + z^2} \right| \leq M$$

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(i) $\left(\frac{1}{t} - i\right)^2 = \frac{1}{z^2} - 1 - \frac{2i}{z}$

$$\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = \int_1^2 \frac{1}{t^2} - 1 \, dt - 2i \int_1^2 \frac{1}{t} dt$$

$$= \left[\frac{-1}{t} - t\right]_1^2 - 2i \Big[\ln|t|\Big]_1^2$$

$$= \left[-\frac{1}{2} - 2\right] - \left[(-1) - 1\right] - 2i\left[\ln(2) - \ln(1)\right]$$

$$= -\frac{1}{2} - 2i\ln(2) = \boxed{-\frac{1}{2} - i\ln(4)}$$

(ii) $$\int_0^{\pi/6} e^{i2t} dt = \int_0^{\pi/6} \cos(2t)\,dt + i\int_0^{\pi/6} \sin(2t)\,dt$$

$$= \left[\frac{1}{2}\sin(2t)\right]_0^{\pi/6} + i\left[-\frac{1}{2}\cos(2t)\right]_0^{\pi/6}$$

$$= \frac{\sqrt{3}}{4} + i\left[-\frac{1}{4} + \frac{1}{2}\right]$$

$$= \boxed{\frac{\sqrt{3}}{4} + \frac{i}{4}}$$

(iii) $\int_0^\infty e^{-zt}\,dt = \lim_{a\to\infty}\int_0^a e^{-zt}\,dt = \lim_{a\to 0}\left[-\frac{e^{-zt}}{z}\right]_0^a$

$$= \lim_{a\to\infty}\left[-\frac{e^{-za}}{z} + \frac{1}{z}\right] = \frac{1}{z}\lim_{a\to\infty}\left[1-e^{-za}\right] = \frac{1}{z}$$

when $Re(z) > 0$. <u>Note</u>: if $Re(z) = 0$ this does not converge.

2. Recall the mean-value theorem from real-valued calculus: If $f(t) : \mathbb{R} \to \mathbb{R}$ is continuous function on the interval $a \le t \le b$ then there exists some $c \in [a, b]$ such that

$$\int_a^b f(t)dt = f(c)(b-a).$$

(i) Does it remain true for $f(t) : \mathbb{R} \to \mathbb{C}$ (yes or no, no proof needed)?

(ii) Try the mean-value theorem out on $f(t) = t + it^2$ for $t \in [0, 1]$ and $f(t) = e^{2it}$ for $t \in [0, \pi]$.

(iii) Explain why the mean-value theorem failed in the above two cases.

(iv) Come up with an (non-constant function) example for which the mean-value theorem holds.

(i) No

(ii) $\int_0^1 t + it^2 \, dt = \frac{1}{2} + \frac{1}{3}i \neq f(c)$ for any $c \in [0,1]$

$$\int_0^\pi e^{2it} dt = \int_0^\pi \cos(2t)dt + i\int_0^\pi \sin(2t)dt$$

$$= \left[\frac{1}{2}\sin(2t)\right]_0^\pi + i\left[-\frac{1}{2}\cos(2t)\right]_0^\pi$$

$$= 0 + i\left[-\frac{1}{2} - \left(\frac{-1}{2}\right)\right] = 0.$$

but $|f(c)|(\pi - 0) = \pi \neq 0$ for all $c \in \mathbb{R}$.

(iii) The theorem fails because it tells us the $\exists c \in [a,b]$ s.t. $\int_a^b Re(f(t))dt = Re(f(c))(b-a)$ and $\exists c' \in [a,b]$ s.t. $i\int_a^b Im(f(t))dt = i\, Im(f(c'))(b-a)$ but c, c' are not guaranteed to be the same.

(iv) if $f(t) = u(t) + iv(t)$ where $u, v : \mathbb{R} \to \mathbb{R}$

Then the mean-value theorem will hold.

take $f(t) = t + it$ for example.

3. Evaluate $\int_\Gamma \bar{z} dz$ from $z = 0$ to $z = 4 + 2i$ along Γ given by

(i) $z = t^2 + it$

(ii) The line from $z = 0$ to $z = 2i$ then along the line $z = 2i$ to $z = 4 + 2i$.

(iii) Why do the above two results not violate the theorem that contour integrals are path independent?

(i) $$\int_\Gamma \bar{z} dz = \int_0^2 (t^2 - it)[2t + i]\, dt$$

$$= \int_0^2 2t^3 + t\, dt + i \int_0^2 t^2 - 2t^2\, dt$$

$$= \left[\frac{1}{2}t^4 + \frac{1}{2}t^2\right]_0^2 + i\left[-\frac{1}{3}t^3\right]_0^2 dt$$

$$= [8 + 2] + i\left[-\frac{8}{3}\right] = \boxed{10 - \frac{8}{3}i}$$

(ii)

$$\int_\Gamma \bar{z}\, dz = i\int_0^2 -it\, dt + \int_0^4 t\, dt - i\int_0^4 2\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^2 + \left[\frac{t^2}{2}\right]_0^4 + i\left[2t\right]_0^4$$

$$= \boxed{10 - 8i}$$

(iii) The issue is that $\bar{z}$ does not have an antiderivative.

4. Integrate $\int_\Gamma z^i dz$ for the principal branch of

$$z^i = e^{i\mathrm{Log} z} \qquad -\pi < \mathrm{Arg} z \leq \pi$$

and Γ the semicircle of unit radius in the upper half plane.

$$\Gamma(t) = e^{it} \qquad t \in [0, \pi]$$

$$\int_\Gamma z^i dz = \int_0^\pi e^{i \log(e^{it})} \cdot ie^{it} d1$$

$$= \int_0^\pi e^{\log(e^{-t})} \cdot ie^{it} dt$$

$$= \int_0^\pi e^{-t} [i\cos(t) - \sin t] dt$$

$$= \frac{1+e^{-\pi}}{2}(i-1) = \frac{i}{i-1}(-e^{-\pi}-1)$$

$$= \frac{(i-1)}{2}(1+e^{-\pi}) = \frac{1}{1+i}(1+e^{-\pi})$$

$$= \frac{i}{1-i}(1+e^{-\pi})$$

$$= \int_0^\pi ie^{-t}e^{it} dt$$

$$= \int_0^\pi ie^{(i-1)t} dt$$

$$= \frac{i}{i-1}\left[e^{(i-1)t}\right]_0^\pi = \frac{i}{i-1}\left[e^{i\pi}e^{-\pi} - 1\right] = \frac{i}{i-1}(-e^{-\pi}-1)$$

5. Take Γ to be the upper half of a circle of radius R. Find M in terms of R such that

$$\left|\int_\Gamma \frac{dz}{2+z^2}\right| \le M$$

$$|2+z^2| \geq ||z^2|-2|$$

$$\left|\int_\Gamma \frac{dz}{2+z^2}\right| \leq \int_\Gamma \left|\frac{1}{2+z^2}\right| dz$$

$$\leq \int_\Gamma \frac{1}{||z^2|-2|} dz \qquad \text{since } |z|=R \text{ on } \Gamma$$

$$= \int_\Gamma \frac{1}{|R^2-2|} dz$$

$$= \boxed{\frac{\pi R}{|R^2-2|}}.$$