Homework #6

1. Take Γ_1 to be the box with side length 2 centered at the origin, and Γ_2 to be the circle with radius 4 centered at the origin. For the following, is it guaranteed that $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$?

   (i) $f(z) = \frac{1}{3z^2+1}$

   (ii) $f(z) = \frac{z+2}{\sin(z/2)}$

   (iii) $f(z) = \frac{z+2}{\sin(z)}$

   (iv) $f(z) = \frac{e^z}{z^2+4}$

   (v) $f(z) = \sqrt{z^2 + \frac{1}{4}}$

2. For the following function
$$f(z) = \frac{z+1}{z^2}$$
   (i) Determine where it is analytic.
   (ii) Compute $\int_C f(z)dz$ where C is a circle of arbitrary radius around the origin by parameterizing C and computing directly.
   (iii) Compute the same integral using the appropriate Cauchy-Integral formula.

3. Let C be the circle $|z - z_0| = R$ traversed counter-clockwise, and let α be any nonzero real number.

   (i) Parametrize C and compute that
$$\oint_C (z - z_0)^{\alpha-1}\, dz = 2iR^\alpha \frac{\sin(\pi\alpha)}{\alpha}$$
   on the principal branch of the integrand.

   (ii) What does this show for $\alpha = n$, integer?

   (iii) In particular, deduce the value for $\alpha = 0$

4. Show that

   (i) $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)d\theta = 2\pi$

   (ii) $\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)d\theta = 0$

   Hint: start with the change of variables $z = e^{i\theta}$. What does the line $\theta \in [0, 2\pi]$ map to under this transformation?

5. Let Γ be the positively oriented boundary of a square whose sides lie along $x = \pm 2$ and $y = \pm 2$. Evaluate:

   (i) $\int_\Gamma \frac{e^{-z}}{z-\pi i/2}dz$

   (ii) $\int_\Gamma \frac{\cosh z}{z^4}dz$

   (iii) $\int_\Gamma \frac{\cos z}{z(z^2+8)}dz$

1. Take Γ_1 to be the box with side length 2 centered at the origin, and Γ_2 to be the circle with radius 4 centered at the origin. For the following, is it guaranteed that $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$?

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   (v) $f(z) = \sqrt{z^2 + \frac{1}{4}}$

(i) Yes, $3z^2+1=0 \iff z = \pm i\frac{1}{\sqrt{3}}$ which is inside Γ_1 and Γ_2

(ii) Yes, $\sin(z/2)=0 \iff z = 2\pi n,\ n\in\mathbb{Z}.$

0 is inside Γ_1 and Γ_2 and $2\pi n$ for $n\in\mathbb{Z}\setminus\{0\}$ is outside Γ_1 and Γ_2.

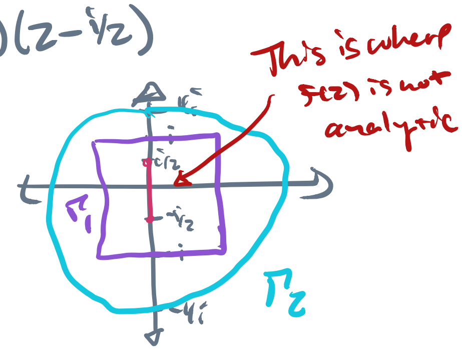
(iii) No, $\sin(z)=0 \iff z=\pi n,\ n\in\mathbb{Z},$

π is inside Γ_2 but not Γ_1.

(iv) No, $z^2+4=0 \iff z=\pm 2i.$

$\pm 2i$ is inside Γ_2 but not Γ_1

(v) Yes, $\sqrt{z^2+1/4} = \sqrt{(z+i/2)(z-i/2)}$



2. For the following function

$$f(z) = \frac{z+1}{z^2}$$

(i) Determine where it is analytic.
(ii) Compute $\int_C f(z)dz$ where C is a circle of arbitrary radius around the origin by parameterizing C and computing directly.
(iii) Compute the same integral using the appropriate Cauchy-Integral formula.

(i) $f(z)$ is analytic as long as $\boxed{z \neq 0.}$

(ii) curve $c = Re^{it}$, $t \in [0, 2\pi]$

$$\int_c f(z)dz = \int_0^{2\pi} \left[\frac{(Re^{it}+1)}{R^2 e^{2it}}\right](iRe^{it})\,dt$$

$$= \int_0^{2\pi} i + ie^{-it}\,dt$$

$$= 2\pi i + \left[-e^{-it}\right]_0^{2\pi} = \boxed{2\pi i}$$

(iii) $g(z) = z+1$, $g'(z) = 1$

$$\int_c f(z)dz = \int_c \frac{g(z)}{(z-0)^2}dz = \frac{2\pi i}{1!}g'(0)$$

$$= 2\pi i(1)$$

$$= \boxed{2\pi i}$$

3. Let C be the circle $|z - z_0| = R$ traversed counter-clockwise, and let α be any nonzero real number.

   (i) Parametrize C and compute that

$$\oint_C (z - z_0)^{\alpha-1}\, dz = 2iR^\alpha \frac{\sin(\pi\alpha)}{\alpha}$$

   on the principal branch of the integrand.

   (ii) What does this show for $\alpha = n$, integer?

   (iii) In particular, deduce the value for $\alpha = 0$

(i) shift everything so that $z_0 = 0$.

The C is parameterized by $\gamma(t) = Re^{it}, t \in [-\pi, \pi]$

and we need to compute

$$\oint_\gamma z^{\alpha-1}\, dz = \int_{-\pi}^{\pi} R^{\alpha-1} e^{i(\alpha-1)t} \cdot Rie^{it}\, dt$$

$$= R^\alpha \int_{-\pi}^{\pi} i e^{i\alpha t}\, dt$$

$$= \frac{R^\alpha}{\alpha} \left[e^{i\alpha t} \right]_{-\pi}^{\pi}$$

$$= \frac{R^\alpha}{\alpha} \left[[\cos(\pi\alpha) - \cos(-\pi\alpha)] + i [\sin(\pi\alpha) - \sin(-\pi\alpha)] \right]$$

$$= \frac{R^\alpha}{\alpha} 2i\sin(\pi\alpha) = \boxed{2iR^\alpha \frac{\sin(\pi\alpha)}{\alpha}}$$

(ii) $\oint_C (z-z_0)^{n-1} = \boxed{0}$ for $n \in \mathbb{Z} \setminus \{0\}$

(iii) Computing directly

$$\oint_\gamma z^{-1} = \int_{-\pi}^{\pi} R^{-1} e^{-it} \cdot Rie^{it} dt$$

$$= i \int_{-\pi}^{\pi} 1\, dt = \boxed{2\pi i}$$

4. Show that

(i) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\pi$

(ii) $\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)d\theta = 0$

Hint: start with the change of variables $z = e^{i\theta}$. What does the line $\theta \in [0, 2\pi]$ map to under this transformation?

Consider the integral

$\oint_C \frac{e^z}{z} dz$ for C the circle of radius 1. By Cauchy Integral formula

$\oint_C \frac{e^z}{z} dz = 2\pi i e^0 = 2\pi i$. Directly computing using the parameterization $z = e^{i\theta}$ $\theta \in [0, 2\pi]$ gives

$$2\pi i = \int_0^{2\pi} \frac{e^{(e^{i\theta})}}{e^{i\theta}} i e^{i\theta} d\theta$$

$$= \int_0^{2\pi} i e^{\cos\theta + i\sin\theta} d\theta$$

$$= i\int_0^{2\pi} e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta)) d\theta$$

$$= \underline{-\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta) d\theta} + i\underline{\int_0^{2\pi} e^{\cos\theta}\cos\sin\theta d\theta}$$

Real part
negative of Integral (ii)

Imaginary part
integral (i)

$\therefore -\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 0$ as desired

Also, $2\pi i = i \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$

$\therefore \quad 2\pi = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$ as desired

5. Let Γ be the positively oriented boundary of a square whose sides lie along $x = \pm 2$ and $y = \pm 2$. Evaluate:

(i) $\int_\Gamma \frac{e^{-z}}{z-\pi i/2} dz$

(ii) $\int_\Gamma \frac{\cosh z}{z^4} dz$

(iii) $\int_\Gamma \frac{\cos z}{z(z^2+8)} dz$

(i)

$$\int_\Gamma \frac{e^{-z}}{z-\pi i/2}\,dz = 2\pi i\, e^{-i\pi/2} = \boxed{2\pi}$$

(ii)

$$\int_\Gamma \frac{\cosh(z)}{z^4}\,dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}\cosh\right](0)$$

$$= \frac{\pi i}{3}\sinh(0) = \boxed{0}$$

(iii)

$$\int_\Gamma \frac{\cos(z)/(z^2+8)}{z}\,dz$$

$$= 2\pi i\left[\frac{\cos(0)}{0^2+8}\right] = \boxed{\frac{\pi i}{4}}$$