Homework #7

1. Let
$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left(z^2-1\right)^n .$$
(i) Show that $P_n(z)$ is a polynomial of order n. These polynomials are called Legendre's polynomials.

(ii) Show that
$$P_n(z) = \frac{1}{2^{n+1}\pi i}\oint_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\,ds, \qquad n = 0, 1, 2, \ldots,$$
where C is any positively-oriented simple closed contour surrounding the point z.

(iii) When $z = 1$, show that the integrand in (ii) can be written as $(s+1)^n/(s-1)$, and deduce that $P_n(1) = 1$. Likewise, calculate that $P_n(-1) = (-1)^n$, $n = 0, 1, 2, \ldots$.

2. Let $f(z)$ be an entire function such that $|f(z)| \leq C|z|$, where C is a positive constant. Deduce that $f(z) = Az$ for some complex constant A.

HINT: Use an appropriate Cauchy's estimate from class to conclude that $f''(z) = 0$ everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $C(|z_0| + R)$.

3. Find the Taylor expansions around the origin of the functions

(i) $\dfrac{\sin z^2}{z^2}$,

(ii) $z \cosh z^2$,

(iii) $\dfrac{z}{z^4+9}$,

and determine their radii of convergence.

4. Find the Taylor series for the function e^z around $z = 1$. Do this without the definition of its coefficients in terms of the derivatives of $f(z)$.

5. (i) Using the definition of its coefficients in terms of the derivatives of $f(z)$ at the center of its circle of convergence, derive the Taylor series around the origin for the function
$$f(z) = \log(1+z),$$
where the principal branch of the logarithm is taken. What is its radius of convergence?

(ii) Use part (i) to find the Taylor series around the origin for the function
$$g(z) = \log\frac{1+z}{1-z},$$
where, again, the principal branch is considered. What is the radius of convergence of this series?

1. Let

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left(z^2 - 1\right)^n .$$

(i) Show that $P_n(z)$ is a polynomial of order n. These polynomials are called Legendre's polynomials. 3pts

(ii) Show that

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\, ds, \qquad n = 0, 1, 2, \ldots,$$ 3pts

where C is any positively-oriented simple closed contour surrounding the point z.

(iii) When $z = 1$, show that the integrand in (ii) can be written as $(s+1)^n/(s-1)$, and deduce that $P_n(1) = 1$. Likewise, calculate that $P_n(-1) = (-1)^n$, $n = 0, 1, 2, \ldots$. 4pts

(i)

$$f(z) = (z^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} (z^2)^k (-1)^{n-k}$$

so $$\frac{d^n}{dz^n} f(z) = \sum_{k=\frac{n}{2}}^{n} \binom{n}{k} \frac{2k!}{(2k-n)!} z^{2k-n} (-1)^{n-k}$$

Hence the highest power is when $k=n$ and gives $z^{2n-n} = z^n$.

(ii) By Cauchy's Formula we get

$$2^n P_n(z) = \frac{1}{n!} \frac{d^n}{dz^n} f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} ds$$

Hence, $$P_n(z) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s^2-1)^n}{(s-z)^{n+1}} ds$$

(iii)

$$P_n(1) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s+1)^n (s-1)^n}{(s-1)^{n+1}}\, ds$$

$$= \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s+1)^n}{(s-1)}\, ds$$

By Cauchy's formula

$$= \frac{1}{2^{n+1}\pi i} \left[2\pi i \ (1+1)^n \right]$$

$$= 1.$$

$$P_n(-1) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s+1)^n (s-1)^n}{(s+1)^{n+1}}\, ds$$

$$= \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s-1)^n}{(s+1)^{n+1}}\, ds$$

By Cauchy's formula.

$$= \frac{1}{2^{n+1}\pi i} \left[2\pi i (-1-1)^n \right] ds$$

$$= (-1)^n. \quad \blacksquare$$

2. Let $f(z)$ be an entire function such that $|f(z)| \le C|z|$, where C is a positive constant. Deduce that $f(z) = Az$ for some complex constant A.

   HINT: Use an appropriate Cauchy's estimate from class to conclude that $f''(z) = 0$ everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $C(|z_0| + R)$.

Let γ be a circle of radius R about z.

$$f''(z) = \frac{2!}{2\pi i} \oint_\gamma \frac{f(s)}{(s-z)} ds$$

$$= \frac{2!}{2\pi i} \oint_{\gamma'} \frac{f(s+z)}{s^3} ds \quad \text{where } \gamma' \text{ is a circle about } 0$$

$$\le \frac{2!}{2\pi i} \oint_{\gamma'} \frac{|s|+|z|}{s^3} ds$$

$$= \frac{2!}{2\pi i} \oint_0^{2\pi} \frac{R+|z|}{R^3} Re^{i\theta} d\theta$$

$$\le \frac{2!}{i}\left[\frac{R+|z|}{R^2}\right] \quad \text{However this holds for all } R$$

Hence

$$|f''(z)| \le \lim_{R\to\infty} \frac{2R+2|z|}{R^2} = 0.$$

$\therefore$ $f''(z) = 0$ and $f(z) = Az$ for some $A \in \mathbb{C}$
since $f(0) \le A\cdot 0 = 0$.

3. Find the Taylor expansions around the origin of the functions

(i) $\dfrac{\sin z^2}{z^2}$,

(ii) $z \cosh z^2$,

(iii) $\dfrac{z}{z^4+9}$,

and determine their radii of convergence.

(i) $$\sin(z^2) = \sum_{n=0}^{\infty} \frac{(z^2)^{2n+1}(-1)^n}{(2n+1)!}$$

$$= z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots$$

So,

$$\frac{\sin(z^2)}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z^2)^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n}}$$

$$= 1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \frac{z^{12}}{7!} + \dots$$

Radius of convergence $= \infty$

Note: $\sin(z^2)/z^2$ is not defined at $z=0$ but the power series is. This gives us a way to extend $\sin(z^2)/z^2$ to be an entire function.

(ii) $z\cosh(z^2)$

$$\cosh(z) = \frac{e^{z^2}+e^{-z^2}}{2}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{z^{4n}}{2n!}$$

so, $$z\cosh(z^2) = \boxed{\sum_{n=0}^{\infty} \frac{z^{4n+1}}{2n!}}$$

$$= x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \cdots$$

Radius of convergence $= \infty$

(iii) $\dfrac{z}{z^4+9} = \dfrac{z/9}{1-(-z^4/9)}$

$$= \frac{z}{9}\sum_{k=0}^{\infty}\left(-z^4/9\right)^k$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k z^{4k}}{9^{k+1}}$$

Need $\left|\dfrac{z^4}{9}\right| \leq 1 \Rightarrow |z| \leq \sqrt{3}$.

Radius of convergence $= \sqrt{3}$

4. Find the Taylor series for the function e^z around $z = 1$. Do this without the definition of its coefficients in terms of the derivatives of $f(z)$.

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$e^z = e^{1+(z-1)} = e^1 e^{z-1} = e^1 \sum_{k=0}^{\infty} \frac{(z-1)^k}{k!}$$

$$= \boxed{\sum_{k=0} \frac{e(z-1)^k}{k!}}$$

5. (i) Using the definition of its coefficients in terms of the derivatives of $f(z)$ at the center of its circle of convergence, derive the Taylor series around the origin for the function

$$f(z) = \log(1+z),$$

where the principal branch of the logarithm is taken. What is its radius of convergence?

(ii) Use part (i) to find the Taylor series around the origin for the function

$$g(z) = \log\frac{1+z}{1-z},$$

where, again, the principal branch is considered. What is the radius of convergence of this series?

(i) $f'(z) = \frac{1}{1+z}$ $\quad f''(z) = \frac{-1}{(1+z)^{-2}}$, $\quad f^{(k)}(z) = \frac{(k-1)!\,(-1)^{k+1}}{(1+z)^{-k}}$

$\Rightarrow$ $f(0)=0$, $\quad f^{(k)}(0) = (-1)^{k+1}(k-1)!$

$\Rightarrow$

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{k!}(z)^k$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k}$$

Radius of convergence is 1.

$$\sum_{k=1}^{\infty} \frac{(-z)^k}{k}$$

$\|$

(ii) $g(z) = f(z) - f(-z)$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k + \sum_{k=1}^{\infty} \frac{(-1)^k}{k}(-z)^k$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{2z^{2k-1}}{2k-1}} = \sum_{k=0}^{\infty} \frac{2z^{2k+1}}{2k+1}$$

Radius of convergence is 1.