Homework #8

1. (i) Find the Laurent Series and region of convergence for $z^2 \sin\left(\frac{1}{z^2}\right)$

   (ii) Find the Laurent Series about $z = 0$ valid for $1 < |z| < \infty$ for $\frac{1}{z(1-z^2)}$

2. Consider
$$f(z) = \frac{1}{(z-1)(z-3)}$$
   (i) Find Laurent Series about $z = 1$ and $z = 3$
   (ii) Find Taylor/Laurent Series about $z = 0$ for all the different regions of convergence.

3. Find the Taylor/Laurent Series about all zeros/singularities of $\frac{z}{1+z^2}$

4. Find the Taylor/Laurent Series about $z = 0$ for
   (i) $\text{arctanh}(z)$
   (ii) $\frac{\text{arctanh}(z)}{1-z^2}$ (if you cannot obtain the full expansion, write out the first 3 non-zero terms)

5. Using Taylor/Laurent series expansions compute the following limits at removable singularities:

   (i) $\lim_{z\to 0} \frac{1}{z}\left[(1+z)^{\frac{1}{z}} - e\right]$,

   (ii) $\lim_{z\to 0} \frac{\sinh z^2 - \text{arctanh } z^2}{z^6}$,

   (iii) $\lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$.

1. (i) Find the Laurent Series and region of convergence for $z^2 \sin\left(\frac{1}{z^2}\right)$

(ii) Find the Laurent Series about $z = 0$ valid for $1 < |z| < \infty$ for $\frac{1}{z(1-z^2)}$

(i) $\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$

$\sin\left(\frac{1}{z^2}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{-4k-2}}{(2k+1)!}$

so,

$z^2 \sin\left(\frac{1}{z^2}\right) = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{z^{-4k}}{(2k+1)!}}$ converges for $z \neq 0$.

(ii) $\frac{1}{z(1-z^2)} = \frac{1}{z}\left(\frac{1/z^2}{1/z^2 - 1}\right)$

$= \frac{-1}{z}\left(\frac{1}{z^2}\right)\left(\frac{1}{1 - 1/z^2}\right)$

$= \frac{-1}{z^3} \sum_{k=0}^{\infty} \left(\frac{1}{z^2}\right)^k$

$= \boxed{\sum_{k=0}^{\infty} -z^{-2k-3}}$

2. Consider

$$f(z) = \frac{1}{(z-1)(z-3)}$$

(i) Find Laurent Series about $z = 1$ and $z = 3$
(ii) Find Taylor/Laurent Series about $z = 0$ for all the different regions of convergence.

(i) $$f(z) = \frac{1}{(z-1)(z-3)} = \left(\frac{1}{z-1}\right)\left(\frac{1}{(z-1)-2}\right)$$

$$= \frac{1}{(z-1)} \frac{-1/2}{1-(z-1)/2}$$

$$= \frac{1}{(z-1)}\left(\frac{-1}{2}\right)\left[\sum_{k=0}^{\infty} \frac{(z-1)^k}{2^k}\right.$$

$$= \boxed{\sum_{k=0}^{\infty} -\frac{(z-1)^{k-1}}{2^{k+1}}}$$

$$f(z) = \left(\frac{1}{z-3}\right)\left(\frac{-1}{1-z}\right) = \left(\frac{1}{z-3}\right)\left(\frac{-1}{-2-(z-3)}\right)$$

$$= \frac{1}{(z-3)} \frac{1/2}{1-\left(-\frac{(z-3)}{2}\right)}$$

$$= \frac{1}{(z-3)}\left(\frac{1}{2}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k (z-3)^k}{2^k}\right.$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k (z-3)^{k-1}}{2^{k+1}}}$$

(ii) $f(z) = \dfrac{1}{(z-1)(z-3)} = \dfrac{A}{(z-1)} + \dfrac{B}{z-3}$

$1 = A(z-3) + B(z-1)$

$A+B=0$

$1 = -3A - B \Rightarrow A = -1/2 \quad B = 1/2$

$f(z) = \dfrac{-1/2}{z-1} + \dfrac{1/2}{z-3}$

We will need 3 different cases:
1) $|z|<1$
2) $1<|z|<3$
3) $3<|z|$

1) $f(z) = \dfrac{1/2}{1-z} + \dfrac{-1/6}{1-z/3}$

$= \frac{1}{2}\sum_{k=0}^{\infty} z^k + \frac{-1}{6}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k$

$= \boxed{\sum_{k=0}^{\infty} \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^{k+1}\right) z^k}$

2) $f(z) = \frac{1}{2}\left(\dfrac{-1/2}{1-1/z}\right) + \left(\dfrac{-1/6}{1-z/3}\right)$

$= \frac{1}{2}\left(\frac{-1}{z}\right)\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k + \frac{-1}{6}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k$

$= \boxed{\frac{-1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^{k+1} + \frac{-1}{6}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k}$

3) $f(z) = \frac{1}{z}\left(\frac{-1/2}{1-1/z}\right) + \frac{1}{z}\left(\frac{1/2}{1-3/z}\right)$

$$= \frac{1}{z}\left(-\frac{1}{2}\right)\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k + \frac{1}{z}\left(\frac{1}{2}\right)\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{1}{2}\left(3^k - 1\right) z^{-k-1}}$$

3. Find the Taylor/Laurent Series about all zeros/singularities of $\frac{z}{1+z^2}$

$$\frac{z}{1+z^2} = \frac{z}{(z+i)(z-i)}$$

zeros are at $z=0$

singularities are at $z=\pm i$.

1) about $z=0$

$$\frac{z}{1+z^2} = \frac{z}{1-(-z^2)} = z\sum_{k=0}^{\infty}(-z^2)^k$$

$$= \boxed{\sum_{k=0}^{\infty}(-1)^k z^{2k+1}}$$

2) about $z=i$

$$\frac{z}{1+z^2} = \frac{(z+i)-i}{(z+i)(z-i)} = \frac{1}{(z-i)} - \frac{i}{(z+i)(z-i)}$$

$$= \left(\frac{1}{z-i}\right)\left(1-\frac{i}{z+i}\right)$$

$$= \left(\frac{1}{z-i}\right)\left(1-\frac{i}{(z-i)+2i}\right)$$

$$= \left(\frac{1}{z-i}\right)\left(1-\frac{1/2}{1-\left(-\frac{(z-i)}{2i}\right)}\right)$$

$$= \frac{1}{(z-i)}\left(1-\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(z-i)^k}{(2i)^k}\right)$$

$$= \boxed{\frac{1}{2(z-i)} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(z-i)^k}{(2i)^{k+1}}}$$

3) $z = -i$

$$\frac{z}{1+z^2} = \frac{(z-i)+i}{(z+i)(z-i)} = \frac{1}{z+i} + \frac{i}{(z+i)(z-i)}$$

$$= \frac{1}{z+i}\left(1 + \frac{i}{(z+i)-2i}\right)$$

$$= \frac{1}{z+i}\left(1 + \frac{-1/2}{1-\frac{(z+i)}{2i}}\right)$$

$$= \frac{1}{z+i}\left(1 - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z+i}{2i}\right)^k\right)$$

$$= \boxed{\frac{1}{2(z+i)} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{(z+i)^k}{(2i)^{k+1}}}$$

4. Find the Taylor/Laurent Series about $z = 0$ for
(i) $\text{arctanh}(z)$
(ii) $\dfrac{\text{arctanh}(z)}{1-z^2}$ (if you cannot obtain the full expansion, write out the first 3 non-zero terms)

(i) $\frac{d}{dz} \text{arctanh}\, z = \frac{1}{1-z^2}$

$$\text{arctanh}(z) = \int \frac{1}{1-z^2} dz$$

$$= \int \sum_{k=0}^{\infty} z^{2k} dz$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}}$$

(ii) $\frac{\text{arctanh}(z)}{1-z^2} = \left(\sum_{n=0}^{\infty} z^{2n}\right)\left(\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}\right)$

$$= (1 + z^2 + z^4 + \dots)\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7}\right)$$

$$= z + \underbrace{\left(1 + \tfrac{1}{3}\right)}_{4/3} z^3 + \underbrace{\left(1 + \tfrac{1}{3} + \tfrac{1}{5}\right)}_{23/15} z^5 + \dots$$

$$= \boxed{\sum_{k=0}^{\infty} \left[\left[\sum_{n=0}^{k} \left(\frac{1}{2n+1}\right)\right] z^{2k+1}\right]}$$

5. Using Taylor/Laurent series expansions compute the following limits at removable singularities:

(i) $\lim_{z\to 0} \frac{1}{z}\left[(1+z)^{\frac{1}{z}} - e\right]$,

(ii) $\lim_{z\to 0} \frac{\sinh z^2 - \text{arctanh } z^2}{z^6}$,

(iii) $\lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$.

$$\frac{1}{z}\log(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^{k-1}}{k}$$

$$(1+z)^{\frac{1}{z}} = e^{1-\frac{z}{2}+\frac{z^2}{3}-\dots}$$

$$(1+z)^{\frac{1}{z}} - e = e\left(e^{\overbrace{-\frac{z}{2}+\frac{z^2}{3}-\dots}^{A(z)}} - 1\right)$$

$$= e\left(1 + A(z) + \frac{A(z)^2}{2!} + \dots - 1\right)$$

$$= e\left(1 + \left(-\frac{z}{2}+\frac{z^2}{3} - \dots\right) + \frac{\left(-\frac{z}{2}+\frac{z^2}{3}-\dots\right)^2}{2!} + \dots - 1\right)$$

$$= e\left(-\frac{z}{2} + \text{higher order terms}\right) \Rightarrow$$

$$\lim_{z\to 0} \frac{(1+z)^{1/z} - e}{z} = \lim_{z\to 0} \frac{e\left(-\frac{z}{2} + \text{higher order terms}\right)}{z}$$

$$= \boxed{-\frac{e}{2}}$$

(ii) $\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$

$$\sinh(z^2) = \sum_{k=0}^{\infty} \frac{z^{4k+2}}{(2k+1)!}$$

$$\text{arctanh}(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}$$

$$\text{arctanh}(z^2) = \sum_{k=0}^{\infty} \frac{z^{4k+2}}{2k+1}$$

$$\frac{\sinh(z^2) - \text{arctanh}(z^2)}{z^6} = \frac{\left(z^2 + \frac{z^6}{3!} + \frac{z^{10}}{5!} \dots\right) - \left(z^2 + \frac{z^6}{3} + \frac{z^{10}}{5} \dots\right)}{z^6}$$

$$= \left(\frac{1}{3!} - \frac{1}{3}\right) + \text{higher order terms}$$

So

$$\lim_{z \to 0} \frac{\sinh(z^2) - \text{arctanh}(z^2)}{z^6} = \boxed{\frac{-1}{6}}$$

iii $$\frac{\sin(z)}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}$$

$$\log\left(\frac{\sin(z)}{z}\right) = \log\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(\sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}\right)^n}{n}$$

$$= \frac{-z^2}{6} + \text{higher order terms}$$

hence

$$\lim_{z\to 0} \frac{1}{z^2} \log\left(\frac{\sin(z)}{z}\right) = -\frac{1}{6}$$

$$\Rightarrow \lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}} = \boxed{e^{-1/6}}$$