Homework #9

Note, we discussed multiple ways to compute the residue at an isolated singularity. I do not tell you which method to use, but suggest trying them all out for practice.

1. What types of singularities do the following functions have and at what points? Are they isolated? Compute the residues at those singularities if applicable. Don't forget about infinity!

   (i) $\dfrac{1}{z(e^z - 1)}$,

   (ii) $\dfrac{z}{z^4 - 1}$,

   (iii) $\sin \dfrac{1}{z^2}$,

   (iv) $\dfrac{\cos z}{z^7}$,

   (v) $\dfrac{e^z - 1}{(\sin z)^3}$,

   (vi) $\dfrac{\sqrt{z}}{1 - z}$.

2. Find the value of the integral

$$\oint_C \frac{3z^2 + 2}{(z-1)(z^2+9)}\, dz,$$

   taken counterclockwise around the circle (i) $|z - 2| = 2$, (ii) $|z| = 4$.

3. Derive that if C is a positively oriented circle $|z| = 8$, then

$$\frac{1}{2\pi i} \oint_C \frac{e^{az}}{\sinh z}\, dz = 1 - 2\cos \pi a + 2\cos 2\pi a.$$

4. Show that

$$\oint_C \frac{dz}{(z^2-1)^2 + 3} = \frac{\pi}{2\sqrt{2}},$$

   where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

1. What types of singularities do the following functions have and at what points? Are they isolated? Compute the residues at those singularities if applicable. Don't forget about infinity!

(i) $\frac{1}{z(e^z-1)}$,

(ii) $\frac{z}{z^4-1}$,

(iii) $\sin\frac{1}{z^2}$,

(iv) $\frac{\cos z}{z^7}$,

(v) $\frac{e^z-1}{(\sin z)^3}$,

(vi) $\frac{\sqrt{z}}{1-z}$.

(i) $\frac{1}{z(e^z-1)} = \frac{1}{z(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots-1)} = \frac{1}{z^2+\frac{z^3}{2!}+\dots}$

$z=0$ is a pole of order 2.

$$\operatorname*{Res}_{z=0} = \lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\frac{z^2}{(z^2+\frac{z^3}{2!}+\dots)} = \lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\frac{1}{1+\frac{z}{2!}+\dots}$$

$$= \lim_{z\to 0}\frac{-1/2}{(1+\frac{z}{2!}+\dots)^2} = \boxed{\frac{-1}{2}}$$

$z=2\pi i k$ $k\neq 0$ is a pole of order 1.

$$\operatorname*{Res}_{z=2\pi i k} = \lim_{z\to 2\pi i k}\frac{z-2\pi i k}{z(e^z-1)}$$

$$= \lim_{z\to 2\pi i k}\frac{1}{ze^z+e^z-1} = \boxed{\frac{1}{2\pi i k}}$$

(ii) Residues at $z = \pm i$ and $z = \pm 1$.

All are simple poles.

$$\operatorname*{Res}_{z=i} = \lim_{z \to i} \frac{(z-i)z}{(z-i)(z+i)(z-1)(z+1)} = \frac{i}{(2i)(i-1)(i+1)}$$

$$= \frac{1}{2(i^2-1)} = \boxed{-1/4}$$

$$\operatorname*{Res}_{z=-i} = \lim_{z \to -i} \frac{(z+i)z}{(z-i)(z+i)(z-1)(z+1)} = \frac{-i}{(-2i)(-i-1)(-i+1)}$$

$$= \frac{1}{2(i^2-1)} = \boxed{-1/4}$$

$$\operatorname*{Res}_{z=1} = \lim_{z \to 1} \frac{(z-1)z}{(z-i)(z+i)(z-1)(z+1)} = \frac{1}{(1-i)(1+i)(2)}$$

$$= \boxed{\frac{1}{4}}$$

$$\operatorname*{Res}_{z=1} = \lim_{z \to -1} \frac{(z+1)z}{(z-i)(z+i)(z-1)(z+1)} = \frac{-1}{(-1-i)(-1+i)(-2)}$$

$$= \boxed{\frac{1}{4}}$$

(iii) $\sin\left(\frac{1}{z^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{z^2}\right)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{-4k-2}}{(2k+1)!}$

essential singularity at $z=0$ with $\operatorname*{Res}_{z=0} = 0$

(iv) $\frac{\cos(z)}{z^7} = \frac{1}{z^7}\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2k!}\right)$

$$= \frac{1}{z^7} - \frac{1}{2z^5} + \frac{1}{24z^3} - \ldots$$

pole of order Z at $z=0$.

$$\operatorname*{Res}_{z=0} = \lim_{z \to 0} \frac{1}{6!} \frac{d^6}{dz^6}\left(\cos(z)\right) = \boxed{\frac{-1}{6!}} = \frac{-1}{720}$$

(v) $\frac{e^{z}-1}{(\sin(z))^3} = \frac{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}{(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots)^3}$

$$= \frac{1+\frac{z}{2!}+\frac{z^2}{3!}+\dots}{z^2(1-\frac{z^2}{3!}+\frac{z^5}{5!}-\dots)^3}$$

$$= \frac{1}{z^2}\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)\left(1-\left(-\frac{z^2}{3!}+\frac{z^5}{5!}-\dots\right)+\left(-\frac{z^2}{3!}+\frac{z^5}{5!}-\dots\right)^2\right)^3$$

$$= \frac{1}{z^2}+\frac{1}{2z}+\text{higher order terms}$$

pole of order 2 at $z=0$

$\underset{z=0}{\text{Res}} = \frac{1}{2}$

$$\frac{e^{z}-1}{(\sin(z))^3} = \frac{(e^{n\pi}-1)+(z-n\pi)+\frac{(z-n\pi)^2}{2}+\dots}{\left((-1)^n(z-n\pi)+(-1)^{n+1}\frac{(z-n\pi)^3}{3!}+\dots\right)^3}$$

$$= \frac{(-1)^n}{(z-n\pi)^3}\left[\frac{(e^{n\pi}-1)+e^{n\pi}(z-n\pi)+e^{n\pi}\frac{(z-n\pi)^2}{2}+\dots}{\left[1-\left(\frac{(z-n\pi)^2}{3!}-\frac{(z-n\pi)^4}{5!}+\dots\right]^3}\right]$$

$$= \frac{(-1)^n}{(z-n\pi)^3}\left[(e^{n\pi}-1)+e^{n\pi}(z-n\pi)+\frac{e^{n\pi}}{2}(z-n\pi)^2+\dots\right]\left[1+s+s^2+\dots\right]^3$$

lowest order term of s is 2 so

$$= \frac{(-1)^{n+1}}{(z-n\pi)^3}\left[(e^{n\pi}-1) + e^{n\pi}(z-n\pi) + \frac{e^{n\pi}}{2}(z-n\pi)^2 + \dots\right]\Big[1 + \tfrac{1}{2}s + \underbrace{s^2 + \dots}_{\text{order is } \geq 4}\Big]$$

$$= \frac{(-1)^{n+1}(e^{n\pi}-1)}{(z-n\pi)^3} + \frac{(-1)^{n+1}e^{n\pi}}{(z-n\pi)^2} + \frac{(-1)^{n+1}(e^{n\pi}-1)}{2(z-n\pi)} + \frac{e^{n\pi}(-1)^{n+1}}{2(z-n\pi)} + \text{h.o.t.}$$

so $\operatorname*{Res}_{z=n\pi} \frac{e^z-1}{\sin(z)^3} = \boxed{(-1)^n e^{n\pi} + \frac{(-1)^{n+1}}{2}}$

Essential singularity at $z=\infty$.

(vi) $\frac{\sqrt{z}}{1-z}$

pole of order 1 at $z=1$, but it is not isolated.

$\operatorname*{Res}_{z\to 1} = \lim_{z\to 1} \frac{(z-1)\sqrt{z}}{1-z} = \lim_{z\to 1} -\sqrt{z} = \boxed{-1}$

There was a lot of confusion over issues at $z=\infty$ so I did not take off points but you should look over the solutions and make sure you understand it.

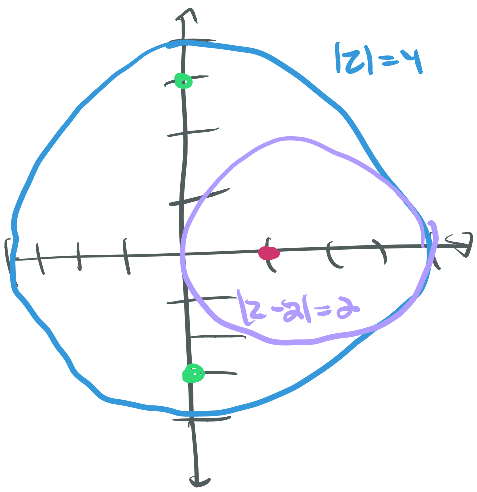
2. Find the value of the integral

$$\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz,$$

taken counterclockwise around the circle (i) $|z-2| = 2$, (ii) $|z| = 4$.

$$f(z) = \frac{3z^2+2}{(z-1)(z^2+9)}$$

singularities at $z=1$ $\quad z = \pm 3i$



All poles are of order 1.

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{3z^2+2}{z^2+9} = \frac{5}{10} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z\to 3i} \frac{3z^2+2}{(z-1)(z+3i)}$$

$$= \frac{-27+2}{(3i-1)(6i)} = \frac{-25}{-18-6i} = \frac{5}{4} - \frac{5i}{12}$$

$$\operatorname*{Res}_{z=-3i} f(z) = \frac{-25}{-18+6i} = \frac{5}{4} + \frac{5i}{12}$$

(i) $$\oint f(z)\,dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = \boxed{\pi i}$$

(ii) $$\oint f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=3i} f(z) + \operatorname*{Res}_{z=-3i} f(z)\right]$$

$$= \pi i + 2\pi i \left(\frac{5}{4} - \frac{5i}{12} + \frac{5}{4} + \frac{5i}{12}\right)$$

$$= \pi i + 5\pi i = \boxed{6\pi i}$$

3. Derive that if C is a positively oriented circle $|z| = 8$, then

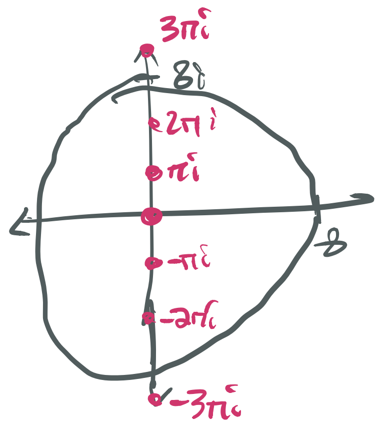
$$\frac{1}{2\pi i}\oint_C \frac{e^{az}}{\sinh z}\,dz = 1 - 2\cos\pi a + 2\cos 2\pi a.$$

$\sinh(z) = i\sin(iz)$ so $\sinh(z) = 0$ when

$z = n\pi i$. we have simple poles.

$$\operatorname*{Res}_{z=n\pi i} \frac{1}{\sinh(z)} = \operatorname*{Res}_{z=n\pi i} \frac{1}{i\sin(iz)} = (-1)^n.$$

So $$\operatorname*{Res}_{z=n\pi i} \frac{e^{az}}{\sinh(z)} = (-1)^n e^{ian\pi}$$
$$= (-1)^n\left(\cos(an\pi) + i\sin(an\pi)\right)$$



$$\frac{1}{2\pi i}\oint_C \frac{e^{az}}{\sinh(z)}\,dz$$

$$= \frac{1}{2\pi i}\cdot 2\pi i\left(\sum_{n=-2}^{2} \operatorname*{Res}_{z=\pi n i} \frac{e^{az}}{\sinh(z)}\right)$$

Remember sin is odd so the imaginary terms cancel and $\sin(0) = 0$. And cos is even.

$$= 1 - 2\cos(\pi a) + 2\cos(2\pi a)$$ ∎

4. Show that

$$\oint_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}},$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

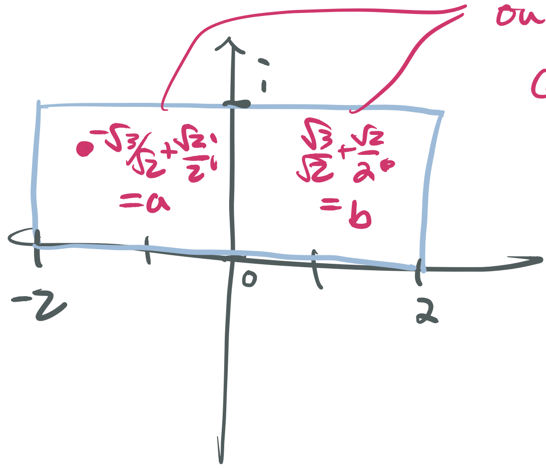
$$f(z) = \frac{1}{(z^2-1)^2+3}$$

$$z^2 - 1 = \pm\sqrt{3}\, i$$

$$z^2 = \pm\sqrt{3}\, i + 1 = 2\left(\pm\frac{\sqrt{3}}{2} i + \frac{1}{2}\right) = 2e^{\frac{\pi}{3} i} \text{ or } 2e^{\frac{5\pi}{3} i}$$

solutions are $z = \pm\sqrt{2}\, e^{\frac{\pi}{6} i}$ or $\pm\sqrt{2}\, e^{\frac{5\pi}{6} i}$

$$z = \overbrace{\frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{2}}{2} i}^{b},\ \overbrace{-\frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{2}}{2} i}^{c},\ \overbrace{-\frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{2}}{2} i}^{a},\ \overbrace{\frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{2}}{2} i}^{d}$$



only these two poles are contained in the curve C.

all poles are of order 1.

$$\operatorname*{Res}_{z=b} f(z) = -\overline{\operatorname*{Res}_{z=a} f(z)}$$

$$\Rightarrow \operatorname*{Res}_{z=a} f(z) + \operatorname*{Res}_{z=b} f(z) = 2\,\mathrm{Im}\left(\operatorname*{Res}_{z=a} f(z)\right) i$$

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} \frac{1}{(z-b)(z-c)(z-d)} = \frac{1}{(a-b)(a-c)(a-d)}$$

$$= \frac{1}{\left(-\frac{2\sqrt{3}}{\sqrt{2}}\right)(\sqrt{2}\, i)\ 2a}$$

$$= \frac{\bar{a}\, i}{8\sqrt{3}} = \left(\frac{-\sqrt{3}}{\sqrt{2}} i - \frac{\sqrt{2}}{2}\right) / 8\sqrt{3}$$

$$= -i/8\sqrt{2} - \sqrt{2}/16\sqrt{3}$$

$$\oint_C f(z)\, dz = 2\pi i \left(\operatorname*{Res}_{z=a} f(z) + \operatorname*{Res}_{z=b} f(z) \right)$$

$$= -4\pi \, \mathrm{Im}\left(\frac{\bar{a}\, i}{8\sqrt{3}}\right)$$

$$= -4\pi \left(\frac{-1}{8\sqrt{2}}\right) = \boxed{\frac{\pi}{2\sqrt{2}}}$$

$$f(z) = \frac{1}{z(e^z - 1)}$$

pole at $z = \infty$ if $f(\frac{1}{z})$ has a pole at $z=0$

and if $\frac{1}{f(\frac{1}{z})}$ has a zero.

$$\frac{1}{f(\frac{1}{z})} = \frac{e^{\frac{1}{z}} - 1}{z} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \dots\right)$$

Solutions to singularities at $z = \infty$

for #1

(i) $\frac{1}{z(e^z-1)}$,
(ii) $\frac{z}{z^4-1}$,
(iii) $\sin\frac{1}{z^2}$,
(iv) $\frac{\cos z}{z^7}$,
(v) $\frac{e^z-1}{(\sin z)^3}$,
(vi) $\frac{\sqrt{z}}{1-z}$.

(i) $\operatorname{Res}_{z=\infty} \frac{1}{z(e^z-1)}$ is undefined since there are an infinite number of poles. $z=\infty$ is not an isolated singular point.

(ii) $\operatorname{Res}_{z=\infty} \frac{z}{z^4-1} = -\operatorname{Res}_{z=0} \frac{1}{z^3(\frac{1}{z^4}-1)}$

$= -\operatorname{Res}_{z=0} \frac{1}{\frac{1}{z}-z^3}$

$= -\operatorname{Res}_{z=0} \frac{z}{1-z^4} = 0.$

$z=\infty$ is isolated and removeable. singularity.

Note: this agrees with the finite singularities

(iii) $\operatorname{Res}_{z=\infty} \sin(\frac{1}{z^2}) = -\operatorname{Res}_{z=0} \frac{1}{z^2}\sin(z^2) = 0.$

$\frac{1}{z^2}\sin(z^2) = 1 - \frac{z^4}{3!} + \ldots$

Note: this agrees with the finite singularities

$z=\infty$ is an isolated and removeable singularity

(iv) $f(z) = \dfrac{\cos(z)}{z^7}$. $\cos(\frac{1}{z})$ has an essential singularity at $z=0$ & $f(z)$ has an isolated essential singularity at at $z=\infty$

$\frac{1}{z^2} f(\frac{1}{z}) = \cos(\frac{1}{z}) z^5 = z^5\left(1 - \frac{z^{-2}}{2} + \frac{z^{-4}}{4!} - \frac{z^{-6}}{6!} + \frac{z^{-8}}{8!} \dots\right)$

$\boxed{\operatorname{Res}_{z=\infty} f(z) = \frac{1}{6!}}$ $z=\infty$ is isolated and removeable

Note: This agrees with the finite singularities.

(v) $\dfrac{e^z - 1}{(\sin(z))^3}$ This has infinite singularities so $\operatorname{Res}_{z=\infty}$ is not defined.

And $z=\infty$ is an essential singularity

(vi) $\dfrac{\sqrt{z}}{z-1}$ This is not defined on the negative real axis so $\operatorname{Res}_{z=\infty}$ is not defined and $z=\infty$ is not an isolated singular point.