Homework #3

1. Find the limit or show it does not exist

   (i) $\lim_{z \to i} \frac{iz^3 - 1}{z + i}$

   (ii) $\lim_{z \to i} \frac{iz^3 - 1}{z - i}$

   (iii) $\lim_{z \to \infty} \frac{(iz-1)^4}{4z^4 - 3i}$

   (iv) $\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2$ (that is z divided by $\bar{z}$)

2. Revisit the property

$$\lim_{z \to z_0} f(z)g(z) = \left(\lim_{z \to z_0} f(z)\right)\left(\lim_{z \to z_0} g(z)\right) = 0$$

   for the case of $\lim_{z \to z_0} g(z) = 0$.

   (i) Provide an example case (for either real-valued or complex-valued functions) when the property holds, and an example for when the property does not hold.

   (ii) Show that the property holds if $|f(z)| < M$ for some positive constant M in the neighborhood of z_0. You may want to return to the delta-epsilon description of the limit.

3. Find the derivative and/or state where the derivative does not exist

   (i) $f(z) = \Re(z)$

   (ii) $f(z) = 0$ if $z = 0$ and $f(z) = \bar{z}^2/z$ otherwise

   (iii) $f(z) = (2z^2 + i)^5$

   (iv) $f(z) = \frac{1}{z}$

   (v) $f(z) = x^2 + iy^2$ for $z = x + iy$

4. Use the definition of a derivative to show the following: Suppose $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist with $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

5. Show that e^z is analytic everywhere (entire).

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Not going to need to prove directly

(i) $\lim_{z\to i} \frac{iz^3-1}{z+i} = \frac{i^4-1}{2i} = \boxed{0}$

(ii) $\lim_{z\to i} \frac{iz^3-1}{z-i} = \frac{i(z^3-i^3)}{z-i} = \frac{i\cancel{(z-i)}(z^2+zi-1)}{\cancel{(z-i)}} = i(-1-1-1) = \boxed{-3i}$

(iii) $\lim_{z\to\infty} \frac{(iz-1)^4}{4z^4-3i} = \lim_{z\to\infty} \frac{(i-1/z)^4}{4-3i/z^4} = \lim_{z\to 0} \frac{(i-z)^4}{4-3iz^4} = \boxed{\frac{1}{4}}$

(iv) If $\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ exists then setting $z=a+bi$ we should get

$\lim_{a\to 0} \left(\frac{a}{\bar{a}}\right)^2 = \lim_{a\to 0} \left(\frac{a}{a}\right)^2 = \boxed{1}$ is the same as

$\lim_{a\to 0} \left(\frac{a+ai}{\overline{a+ai}}\right)^2 = \lim_{a\to 0} \left(\frac{a+ai}{a-ai}\right)^2 = \lim_{a\to 0} \frac{a^2+2a^2i-a^2}{a^2-2a^2i-a^2}$

$= \lim_{a\to 0} \frac{-2a^2i}{2a^2i} = \boxed{-1}$. Since these are not equal $\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ does not exist.

2. Revisit the property

$$\lim_{z \to z_0} f(z)g(z) = \left(\lim_{z \to z_0} f(z)\right)\left(\lim_{z \to z_0} g(z)\right) = 0$$

for the case of $\lim_{z \to z_0} g(z) = 0$.

(i) Provide an example case (for either real-valued or complex-valued functions) when the property holds, and an example for when the property does not hold.

(ii) Show that the property holds if $|f(z)| < M$ for some positive constant M in the neighborhood of z_0. You may want to return to the delta-epsilon description of the limit.

(i) There are a wide range of examples that would work.

For example:

$f(z) = \frac{1}{z}$, $g(z) = z^2$

$\lim_{z \to 0} f(z)g(z) = 0$ while

$f(z) = \frac{1}{z^2}$, $g(z) = z$

$\lim_{z \to 0} f(z)g(z) \neq 0$

(ii) By assumption $\exists \delta' > 0$ such that $|f(z)| < M$ if $|z - z_0| < \delta'$ so We know that $\lim_{z \to z_0} g(z) = 0$

Choose $\delta' > \delta > 0$ s.t. for $|z - z_0| < \delta$, $|g(z) - 0| < \varepsilon / M$. Then for $|z - z_0| < \delta$,

$|f(z)g(z) - 0| \leq |f(z)||g(z) - 0| \leq M|g(z) - 0| < M\varepsilon/M = \varepsilon$.

$\blacksquare$

3. Find the derivative and/or state where the derivative does not exist

(i) $f(z) = \Re(z)$

(ii) $f(z) = 0$ if $z = 0$ and $f(z) = \bar{z}^2/z$ otherwise

(iii) $f(z) = (2z^2 + i)^5$

(iv) $f(z) = \frac{1}{z}$

(v) $f(z) = x^2 + iy^2$ for $z = x + iy$

let $z = x+iy$, $h = a+bi$

(i) $$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{a}{a+bi}$$

this limit does not exist because

$$\lim_{(a,0)\to(0,0)} \frac{a}{a+bi} = 1 \neq 0 = \lim_{(0,b)\to(0,0)} \frac{a}{a+bi}.$$

we could also use the Cauchy-Riemann equations

(ii) let $z = x+iy$. Then

$$f(z) = \frac{(\bar{z})^3}{|z|^2} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3 - 3x^2 iy + 3x(iy)^2 - (iy)^3}{x^2+y^2}$$

$$= \underbrace{\left(\frac{x^3 - 3xy^2}{x^2+y^2}\right)}_{u(x,y)} + \underbrace{\left(\frac{y^3 - 3x^2y}{x^2+y^2}\right)}_{v(x,y)} i$$

$$\frac{\partial u}{\partial x} = \frac{(3x^2 - 3y^2)(x^2+y^2) - (x^3 - 3xy^2)(2x)}{(x^2+y^2)^2} = \frac{3x^4 - 3y^4 - 2x^4 + 6x^2y^2}{x^2+y^2}$$

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$$\frac{\partial v}{\partial y} = \frac{(3y^2 - 3x^2)(x^2+y^2) - (y^3 - 3x^2y)(2y)}{x^2+y^2} = \frac{3y^4 - 3x^4 - 2y^4 + 6x^2y^2}{x^2+y^2}$$

unless $x=y$.

$$\frac{\partial u}{\partial y} = \frac{-\cancel{3}6xy(x^2+y^2)-(2y)(x^3-3xy^2)}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{-6xy(x^2+y^2)-(2x)(y^3-3x^2y^2)}{(x^2+y^2)^2}$$

so $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ when $x=-y$.

This leaves the only possible point where $f(z)$ can be differentiable is at $x=y=-y=0$. Except we have to be more careful at 0 since u,v are not necessarily continuously differentiable at 0. So we need to use the definition.

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{\bar{h}^2/h - 0}{h} = \lim_{h\to 0} \frac{\bar{h}^2}{h^2}$$

$$= \lim_{(a,b)\to(0,0)} \frac{a^2 - 2abi - b^2}{a^2 + 2abi - b^2}$$ which does not exists.

compare $\lim_{(a,a)\to(0,0)}$ and $\lim_{(a,0)\to(0,0)}$.

(iii) Since the usual differentiation rules hold $f'(z) = 5(2z^2+i)^4(4z) = \boxed{20z(2z^2+i)^4}$

<u>Note:</u>

We know this is differentiatable because $g(z)=z$ and $h(z)=z_0$ for $z_0 \in \mathbb{C}$ a constant are differentiable. Then, we can use chain rule/product/sum of function to get $f(z)$ is differentiable.

(iv) $$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{\frac{1}{z+h} - \frac{1}{z}}{h} = \lim_{h\to 0} \frac{z-(z+h)}{h(z+h)z}$$

$$= \lim_{h\to 0} \frac{-h}{h(z+h)z} = \boxed{\frac{-1}{z^2}}$$

(v) $f(x+iy) = \underbrace{x^2}_{u} + i\underbrace{y^2}_{v}$

$$\frac{\partial u}{\partial x} = 2x \neq 2y = \frac{\partial v}{\partial y} \quad \text{unless} \quad x = y$$

$$\frac{\partial u}{\partial y} = 0 = 0 = -\frac{\partial v}{\partial x}.$$

so $f'(x+iy) = 2x + 2yi$ on $x = y$

but does not exist when $x \neq y$.

4. Use the definition of a derivative to show the following: Suppose $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist with $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{z \to z_0} \frac{\dfrac{f(z)-f(z_0)}{|z-z_0|}}{\dfrac{g(z)-g(z_0)}{|z-z_0|}} = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)}$$

$$= \lim_{z \to z_0} \frac{f(z)}{g(z)}. \quad \blacksquare$$

since $f(z_0) = g(z_0) = 0$.

5. Show that e^z is analytic everywhere (entire).

e^z is analytic everywhere if it is differentiable at every point in $\mathbb{C}$. Let $z = x+iy$

$$e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

$$= \underbrace{e^x\cos(y)}_{u} + i\,\underbrace{e^x\sin(y)}_{v}$$

$$\frac{\partial u}{\partial x} = e^x\cos(y) = e^x\cos(y) = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x\sin(y) = -e^x\sin(y) = \frac{\partial v}{\partial x}$$

e^z satisfies the Cauchy-Riemann equations everywhere so it is entire.