Homework #4

1. Compute all the values of

   (i) $\log(e)$

   (ii) $\log(-ei)$

   (iii) $(1+i)^i$

2. Show that, when restricting to values on the principal branch of the logarithm, defined by $-\pi < \arg z < \pi$,

   (i) $\log(1+i)^2 = 2\log(1+i)$,

   (ii) $\log(-1+i)^2 \neq 2\log(-1+i)$.

3. Is it always true that $\text{Log}(z_1 z_2) = \text{Log} z_1 + \text{Log} z_2$? If not, modify this statement so that it is true.

4. Determine where the function is analytic:

   $f(z) = \text{Log}(z-i)$

   $f(z) = \text{Log}(z+4)/(z^2+i)$

   Identify the branch point(s) of the above two functions.

5. Determine where each of the following are analytic: $\cos z$, $\sin z$, $\cos \bar{z}$ and $\sin \bar{z}$

6. Define
$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

   (i) Sketch $\cosh x$ and $\sinh x$ for real x.

   (ii) Derive the addition formulas for $\cosh(z+w)$ and $\sinh(z+w)$, and also the formula $\cosh^2 z - \sinh^2 z = 1$.

   (iii) What kind of a curve does the parametrization $\xi = \cosh t$, $\eta = \sinh t$, $-\infty < t < \infty$ represent in the (ξ, η)-plane?

   (iv) Find the derivatives of $\cosh z$ and $\sinh z$.

   (v) Derive the formulas $\cos z = \cosh iz$ and $\sin z = -i \sinh iz$.

   (vi) Discuss the periodicity $\sinh z$ if it exists, and find all the solutions to $\sinh z = 0$

1. Compute all the values of

(i) $\log(e)$

(ii) $\log(-ei)$

(iii) $(1+i)^i$

(i) $\log(e) = \ln(e) + i(0+2n\pi) = 1 + 2n\pi i, \quad n \in \mathbb{Z}$

(ii) $\log(-ei) = \log(e\, e^{-i\pi/2})$

$= \ln(e) + i(-\frac{\pi}{2} + 2n\pi)$

$= 1 - \frac{\pi}{2}i + 2n\pi i, \quad n \in \mathbb{Z},$

(iii) $(1+i)^i = \exp(i \log(1+i))$

$= \exp\{i[\ln\sqrt{2} + i(\frac{\pi}{4} + 2n\pi)]\}$

$= \exp[\frac{i}{2}\ln 2 - (\frac{\pi}{4} + 2n\pi)]$

$= \exp(-\frac{\pi}{4} - 2n\pi) \exp(\frac{i}{2}\ln(2))$

$= \exp(-\frac{\pi}{4} + 2n\pi) \exp(\frac{i}{2}\ln(2)).$

$n \in \mathbb{Z}.$

2. Show that, when restricting to values on the principal branch of the logarithm, defined by $-\pi < \arg z < \pi$,

(i) $\log(1+i)^2 = 2\log(1+i)$,

(ii) $\log(-1+i)^2 \neq 2\log(-1+i)$.

(i) $\log[(1+i)^2] = \log(2i) = \ln(2) + \frac{\pi}{2}i$

$\parallel$

$2\log(1+i) = 2(\ln\sqrt{2} + i\frac{\pi}{4}) = \ln 2 + \frac{\pi}{2}i$

(ii) $\log[(-1+i)^2] = \log(-2i) = \ln(2) - \frac{\pi}{2}i$

$\nparallel$

$2\log(-1+i) = 2(\ln\sqrt{2} + i\frac{3\pi}{4}) = \ln(2) + \frac{3\pi}{2}i$

3. Is it always true that $\text{Log}(z_1 z_2) = \text{Log} z_1 + \text{Log} z_2$? If not, modify this statement so that it is true.

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. Then

$$\text{Log}(z_1 z_2) = \ln(|z_1 z_2|) + i \arg(z_1 z_2)$$

$$\begin{aligned}
\text{Log}(z_1) + \text{Log}(z_2) &= \ln(|z_1|) + i \arg(z_1) + \ln(|z_2|) + i\arg(z_2) \\
&= \ln(|z_1|) + \ln(|z_2|) + i[\arg(z_1) + \arg(z_2)] \\
&= \text{Log}(z_1) + \text{Log}(z_2)
\end{aligned}$$

if and only if $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Hence, this doesn't always hold.

For example, $z_1 = -1+i$, $z_2 = (-1+i)$. since

$\arg(z_1) = \frac{3\pi}{4} = \arg(z_2)$.

This will always be true if $-\theta < \arg(z_1) + \arg(z_2) < \theta$.

4. Determine where the function is analytic:

$f(z) = \operatorname{Log}(z - i)$

$f(z) = \operatorname{Log}(z + 4)/(z^2 + i)$

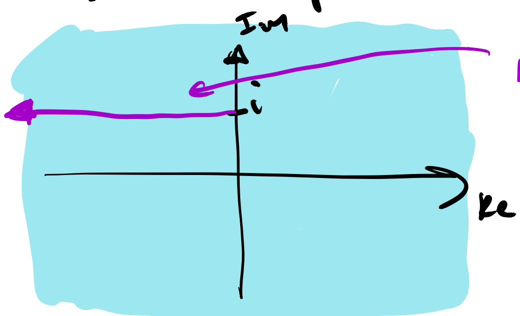
Identify the branch point(s) of the above two functions.

$\log(z)$ has branch points at 0

so, $f(z) = \log(z-i)$ has branch points at $\boxed{z=i \text{ and } z=\infty}$

$f(z)$ is analytic except at the branch cut.

$f(z)$ is analytic on $\mathbb{C} \setminus \{z \in \mathbb{C} : Re(z) \leq 0 \text{ and } Im(z) = i\}$



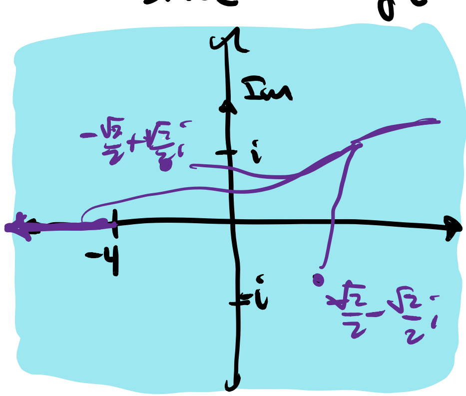
Analytic everywhere except along this line.

• $\log(z+4)$ has a branch point at $\boxed{z=-4 \text{ and } z=\infty}$

• $\frac{1}{z^2+i}$ has a discontinuity when $z^2+i=0$

so at $z = e^{\frac{3\pi}{4}i}, e^{\frac{7\pi}{4}i}$, it is not analytic.

• Since $\log(\pm i+4) \neq 0$ the function is still not analytic there



Analytic everywhere except this line and the points at $z = e^{\frac{3\pi}{4}i}, e^{\frac{7\pi}{4}i}$

• $\frac{1}{z^2}$ has no branch points so $\frac{1}{z^2+i}$ does not.

Branch points at ∞?

Does $f(\frac{1}{z}) = \log(\frac{1}{z} - i)$ have a branch point at $z=0$?

$$\log(\tfrac{1}{z} - i) = \log\left(\frac{1-iz}{z}\right)$$

$$= \log(1-iz) - \log(z)$$

Near $z=0$ this is 0 but we do have issues with the $\log(z)$ term.

So $\log(z-i)$ has a branch point at $z=\infty$.

Does $f(\frac{1}{z}) = \dfrac{\log(\frac{1}{z}+4)}{\frac{1}{z^2}+i}$ have a branch point at $z=0$?

$$f(\tfrac{1}{z}) = \frac{z^2 \log\left(\frac{1+4z}{z}\right)}{1+iz^2}$$

$$= \left(\frac{z^2}{1+iz^2}\right)\left[\log(1+4z) - \log(z)\right]$$

No issue with this term but this term gives us a branch point at $z=0$.

Hence $f(z)$ has a branch point at $z=\infty$.

5. Determine where each of the following are analytic: $\cos z$, $\sin z$, $\cos \bar{z}$ and $\sin \bar{z}$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos(\bar{z}) = \frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2} \qquad \sin(\bar{z}) = \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i}$$

Since e^{iz} is entire and the sum of entire functions is entire so are $\boxed{\sin(z) \text{ and } \cos(z). \text{ are entire}}$

Let $z = x+iy$.

$$e^{i\bar{z}} = e^{i(x-iy)} = e^{y}\cos(x) + i e^{y}\sin(x)$$

$$e^{-i\bar{z}} = e^{-i(x-iy)} = e^{-y}\cos(-x) + i e^{-y}\sin(-x)$$

$$\cos(\bar{z}) = \underbrace{\frac{e^{y}\cos(x) + e^{-y}\cos(-x)}{2}}_{u} + i\left[\underbrace{\frac{e^{y}\sin(x) + e^{-y}\sin(-x)}{2}}_{v}\right]$$

$$u_x = \frac{-e^{y}\sin(x) + e^{-y}\sin(-x)}{2} \qquad v_y = \frac{e^{y}\sin(x) - e^{-y}\sin(-x)}{2}$$

$$u_y = \frac{e^{y}\cos(x) - e^{-y}\cos(-x)}{2} \qquad v_x = \frac{e^{y}\cos(x) - e^{-y}\cos(-x)}{2}$$

$u_x = v_y \iff x = \pi k,\ k \in \mathbb{Z}$

$u_y = -v_x \iff y = 0$ or $x = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$.

So, $\cos(\bar{z})$ is differentiable at

$x = \pi k$, $k \in \mathbb{Z}$.

These are isolated points so $\cos(\bar{z})$ is nowhere analytic.

Let $z = x+iy$. $\cos(z)$.

$e^{i\bar{z}} = e^{i(x-iy)} = e^{y}\cos(x) + i e^{y}\sin(x)$

$e^{-i\bar{z}} = e^{-i(x+iy)} = e^{-y}\cos(-x) + i e^{-y}\sin(-x)$

$$\sin(\bar{z}) = \frac{e^{y}\cos(x) - e^{-y}\cos(-x)}{2i} + i\,\frac{\left(e^{y}\sin(x) - e^{y}\sin(-x)\right)}{2i}$$

$$= i\left[\frac{-e^{y}\cos(x) + e^{-y}\cos(-x)}{2}\right] + \left[\frac{e^{y}\sin(x) - e^{-y}\sin(-x)}{2}\right]$$

$$u_x = \frac{e^{y}\cos(x) + e^{-y}\cos(-x)}{2} \qquad v_y = -\frac{e^{y}\cos(x) - e^{-y}\cos(-x)}{2}$$

$$u_y = \frac{e^{y}\sin(x) + e^{-y}\sin(-x)}{2} \qquad v_x = \frac{e^{y}\sin(x) + e^{-y}\sin(-x)}{2}$$

$u_x = \frac{e^{y}+e^{-y}}{2}\cos(x)$, $v_y = -\left[\frac{e^{y}+e^{-y}}{2}\right]\cos(x)$

$u_x = v_y \iff x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$.

$u_y = \frac{e^{y}-e^{-y}}{2}\sin(x)$, $v_x = -\left[\frac{e^{y}-e^{-y}}{2}\right]\sin(x)$

$u_y = v_x \iff y = 0$ or $x = \pi k$, $k \in \mathbb{Z}$.

So $\sin(\bar{z})$ is differentiable at $z = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

these are isolated points. so $\underline{\sin(\bar{z})}$

$\underline{\text{is nowhere analytic}}$

6. Define

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

(i) Sketch $\cosh x$ and $\sinh x$ for real x.

(ii) Derive the addition formulas for $\cosh(z+w)$ and $\sinh(z+w)$, and also the formula $\cosh^2 z - \sinh^2 z = 1$.

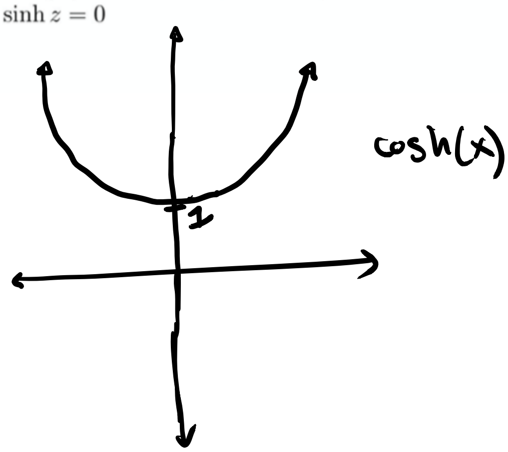
(iii) What kind of a curve does the parametrization $\xi = \cosh t$, $\eta = \sinh t$, $-\infty < t < \infty$ represent in the (ξ, η)-plane?

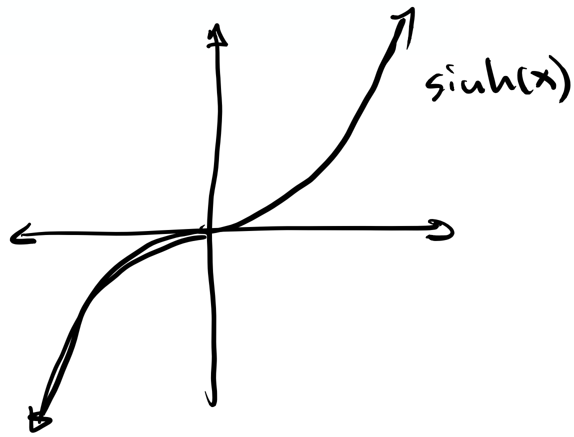
(iv) Find the derivatives of $\cosh z$ and $\sinh z$.

(v) Derive the formulas $\cos z = \cosh iz$ and $\sin z = -i \sinh iz$.

(vi) Discuss the periodicity $\sinh z$ if it exists, and find all the solutions to $\sinh z = 0$

(i)





(ii) $\cosh(z)\cosh(w) + \sinh(z)\sinh(w)$

$$= \left(\frac{e^z + e^{-z}}{2}\right)\left(\frac{e^w + e^{-w}}{2}\right) + \left(\frac{e^z - e^{-z}}{2}\right)\left(\frac{e^w - e^{-w}}{2}\right)$$

$$= \frac{e^z e^w + e^{-z} e^{-w}}{4} + \frac{e^z e^{-w} + e^w e^{-z}}{4}$$

$$+ \frac{e^z e^w + e^{-z} e^{-w}}{4} - \left(\frac{e^z e^{-w} + e^w e^{-z}}{4}\right)$$

$$= \frac{e^z e^w + e^{-z} e^{w}}{2} = \cosh(z+w)$$

In a similiar manner you get

$\sinh(z+w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$.

$$\cosh(z)^2 - \sinh(z)^2 = \frac{(e^z+e^{-z})^2}{4} - \frac{(e^z-e^{-z})^2}{4}$$

$$= \frac{e^{2z}+2+e^{-2z}}{4} - \left(\frac{e^{2z}-2+e^{-2z}}{4}\right)$$

$$= 1.$$

(iii) From (ii) $\xi^2 - \eta^2 = 1$ hence the graph is a hyperbola.

(iv) $\frac{d}{dz}\cosh(z) = \frac{d}{dz}\left[\frac{e^z+e^{-z}}{2}\right] = \frac{e^z-e^{-z}}{2} = \sinh(z)$

$\frac{d}{dz}\sinh(z) = \frac{d}{dz}\left[\frac{e^z-e^{-z}}{2}\right] = \frac{e^z+e^{-z}}{2} = \cosh(z)$

(v) $\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = \cosh(iz)$

$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = -i\left[\frac{e^{iz}-e^{-iz}}{2}\right] = -i\sinh(iz)$.

(vi) from (v) $\cosh(z) = \cos(-iz)$

and $\sinh(z) = i\sin(-iz)$

Since $\cos(z)$ and $\sin(z)$ are periodic with period 2π, sinh and cosh are periodic with period $2\pi i$.

We want to find z so that

$0 = \sinh(z) = i\sin(-iz)$.

$i\sin(-iz) = 0$ when $-iz = \pi k$, $k \in \mathbb{Z}$

so $z = i\pi k$, $k \in \mathbb{Z}$.