13.1 and 13.2: Vectors

May 28, 2020

Big Picture for today: 13.1 and 13.2

Big Picture: (1) Read the syllabus.
(2) vectors = length and direction = points in space.
(3) vector length can be used as *radius* for a sphere.
(4) vectors satisfy certain properties

Welcome and details

Welcome to math 233!

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- Instructor: Derrick Nowak
- Office hours
- Email: dtn7@live.unc.edu
- Website: sakai.unc.edu
- Text: Calculus: Early Transcendentals 3rd Edition, by Briggs, Cochran, Gillett, and Schulz,
- Homework: MyLab (course ID nowak14335) and weekly worksheets in recitation
- Midterms: 2, see syllabus for dates
- Grading: see syllabus

Let P, Q be points in $\mathbb{R}^2$. The *vector* from P to Q is written $\overrightarrow{PQ}$. It has *length* and *direction*. P is the *tail* and Q is the *head*.

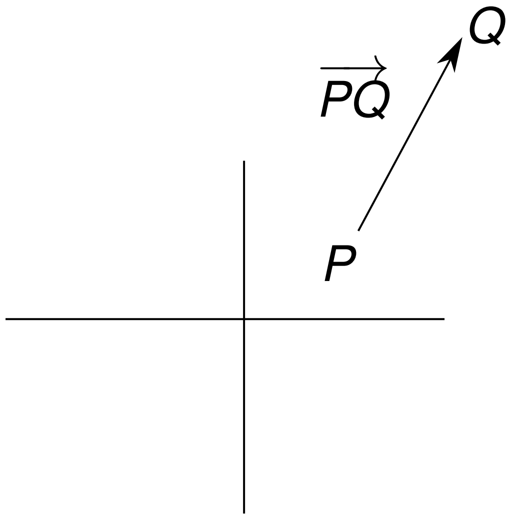

Figure: Vector from P to Q .

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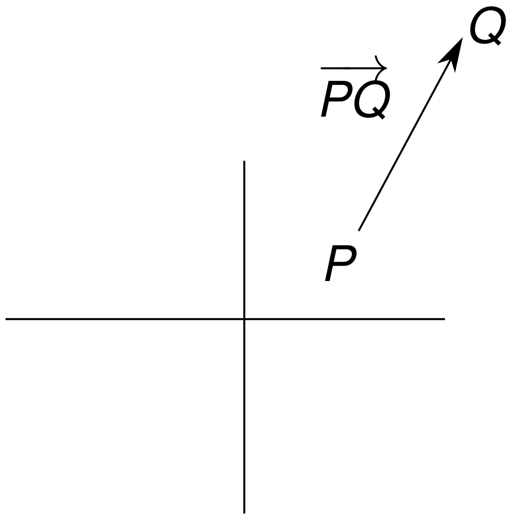

Figure: Vector from P to Q .

Note: "direction" and "length" are *independent of the tail*.

Points as vectors

If (x_1, y_1) is a point in $\mathbb{R}^2$, $\langle x_1, y_1 \rangle$ is the vector from $(0, 0)$ to (x_1, y_1). Use (,) to denote points and < , > to denote vectors

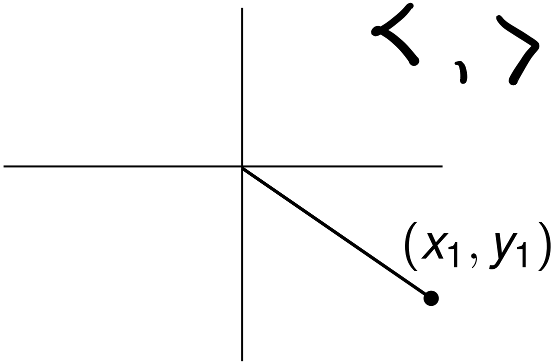

Figure: Vector from $(0, 0)$ to (x_1, y_1) .

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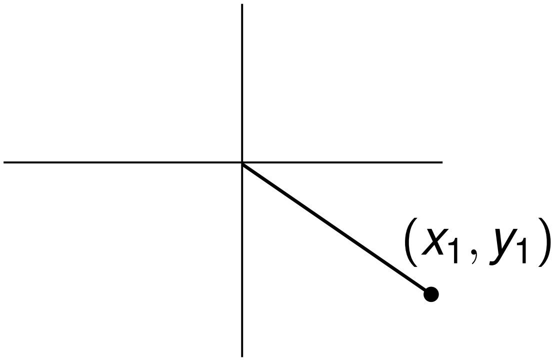

Figure: Vector from $(0, 0)$ to (x_1, y_1) .

Identify points with vectors (called *position vectors*).

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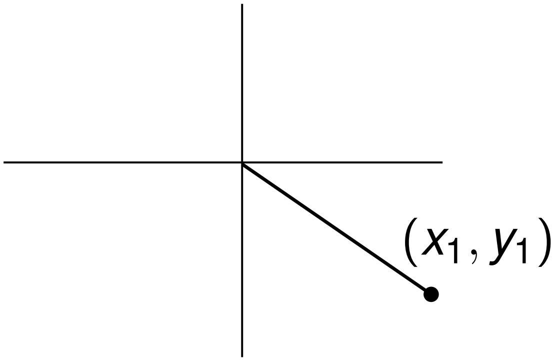

Figure: Vector from $(0,0)$ to (x_1, y_1) .

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Other notations: $\mathbf{u}$, $\mathbf{v}$, etc. and $\vec{u}$, $\vec{v}$, etc. are used to denote vectors.

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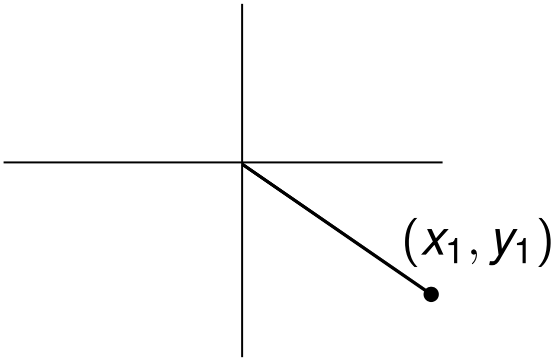

Figure: Vector from $(0,0)$ to (x_1, y_1) .

Identify points with vectors (called *position vectors*).
Other notations: **u**, **v**, etc. and $\vec{u}$, $\vec{v}$, etc. are used to denote vectors. **Question:** What is $\langle 0, 0 \rangle$? This is the zerovector has no direction and 0 length

Scalar multiplication

Scalar multiplication: If $\langle x_1, y_1 \rangle$ is a position vector and a is a real number,

$$a \langle x_1, y_1 \rangle = \langle ax_1, ay_1 \rangle .$$

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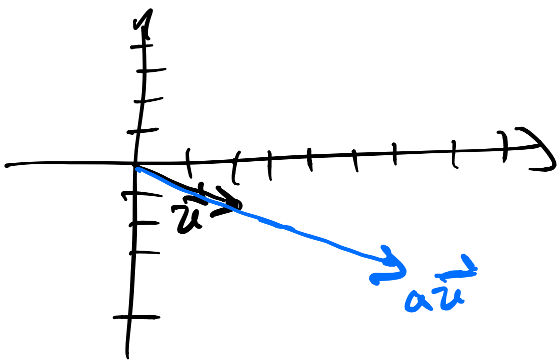
Example: Let $\mathbf{u} = \langle 2, -1 \rangle$ and $a = 3$. Compute and sketch $\mathbf{u}$ and $a\mathbf{u}$.

$a\vec{u} = 3\langle 2,-1\rangle = \langle 3\cdot 2, 3\cdot -1\rangle = \langle 6,-3\rangle$



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Scalar multiplication only changes length

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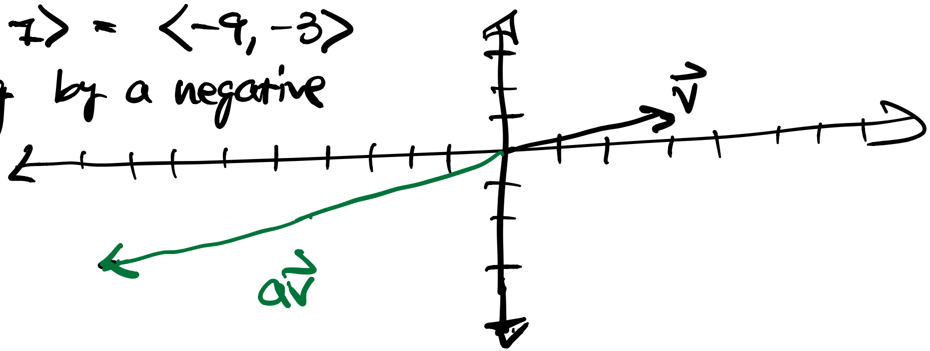
Exercise: Let $\mathbf{v} = \langle 3, 1\rangle$ and $a = -3$ compute and sketch $a\mathbf{v}$.

$a\vec{v} = -3\langle 3,1\rangle = \langle -9,-3\rangle$

note multiplying by a negative changes the direction to the opposite way



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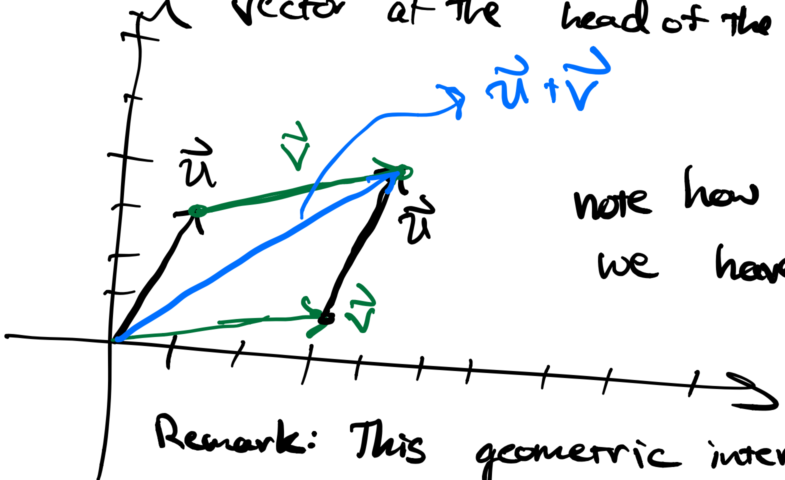
Question: why is it the same direction? **Answer:** trig!
Exercise: Let $\mathbf{v} = \langle 3, 1 \rangle$ and $a = -3$ compute and sketch $a\mathbf{v}$.
Definition: Two position vectors $\mathbf{u}$ and $\mathbf{v}$ are *parallel* if there exists a real number a such that $\mathbf{u} = a\mathbf{v}$.

Vector addition

Let $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$. Then

$$\mathbf{u} + \mathbf{v} = \langle x_1 + x_2, y_1 + y_2 \rangle .$$

we can think of adding vectors by placing the tail of one vector at the head of the other



note how the order doesn't matter.
we have $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

Remark: This geometric interpretation is called the triangle or parallelogram rule.

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Example: Let $\mathbf{w} = \langle 2, -1 \rangle$. Can write

$$\mathbf{w} = \langle 2, 0 \rangle + \langle 0, -1 \rangle$$

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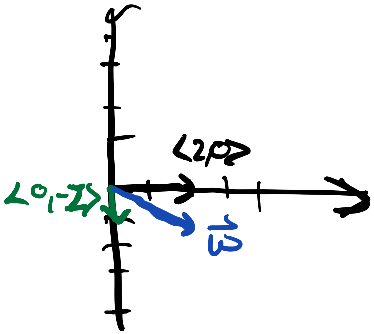
"Go 2 in the x direction, followed by -1 in the y direction."



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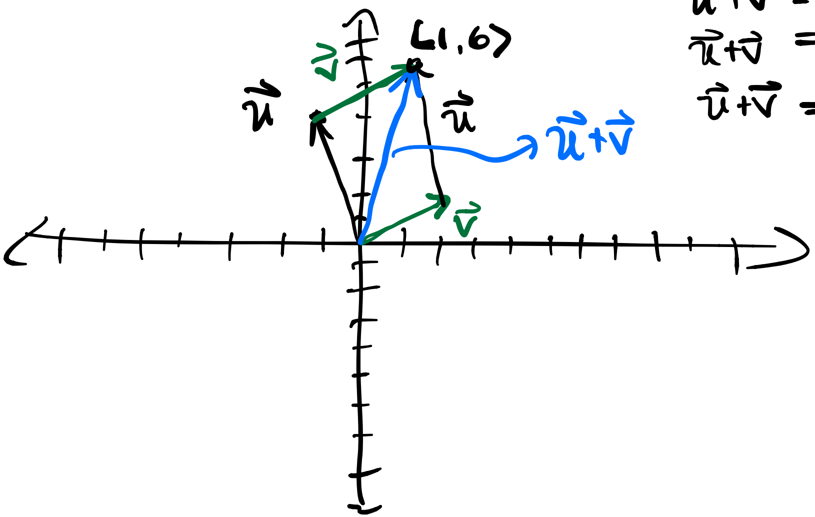
Then

$$\mathbf{u} + \mathbf{v} = \langle x_1, y_1 \rangle + \langle x_2, 0 \rangle + \langle 0, y_2 \rangle$$

says "start at $\mathbf{u}$. Then go x_2 in the x direction. Then go y_2 in the y direction."

Example: Let $\mathbf{u} = \langle -1, 4 \rangle$ and $\mathbf{v} = \langle 2, 2 \rangle$. Compute $\mathbf{u} + \mathbf{v}$ and sketch.



$\vec{u}+\vec{v} = \langle -1,4 \rangle + \langle 2,2 \rangle$

$\vec{u}+\vec{v} = \langle -1+2, 2+4 \rangle$

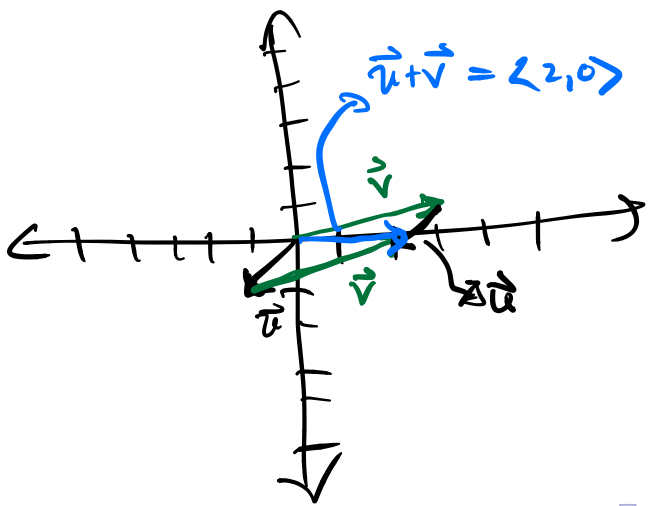
$\vec{u}+\vec{v} = \boxed{\langle 1,6 \rangle}$

Examples

Example: Let $\mathbf{u} = \langle -1, 4 \rangle$ and $\mathbf{v} = \langle 2, 2 \rangle$. Compute $\mathbf{u} + \mathbf{v}$ and sketch.

Exercise: Let $\mathbf{u} = \langle -1, -1 \rangle$ and $\mathbf{v} = \langle 3, 1 \rangle$. Compute $\mathbf{u} + \mathbf{v}$ and sketch.

$$\vec{u} + \vec{v} = \langle -1, -1 \rangle + \langle 3, 1 \rangle$$
$$= \langle -1+3, -1+1 \rangle$$
$$= \langle 2, 0 \rangle$$



Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points in $\mathbb{R}^2$. Let $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$ be the position vectors of P and Q. Then

$$\langle x_1, y_1 \rangle + \overrightarrow{PQ} = \langle x_2, y_2 \rangle$$

$$\rightsquigarrow \overrightarrow{PQ} = \mathbf{u} - \mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle .$$

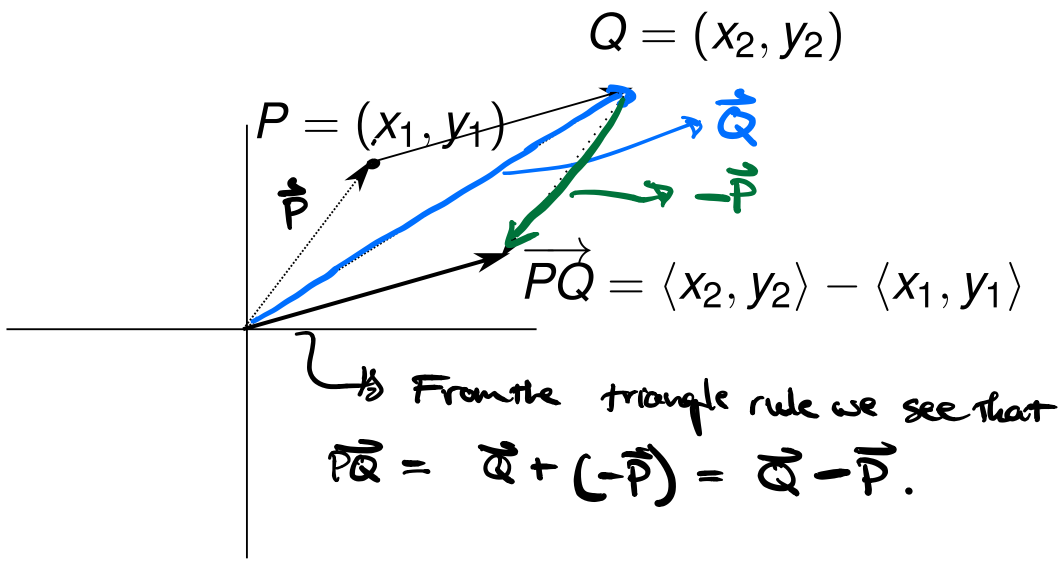


Figure: Vector $\overrightarrow{PQ}$ as position vector.

Vector length

Since $\mathbf{u} = \langle x_1, y_1 \rangle = \langle x_1, 0 \rangle + \langle 0, y_1 \rangle$, length is by Pythagorean formula:

$$|\mathbf{u}| = \sqrt{x_1^2 + y_1^2}.$$

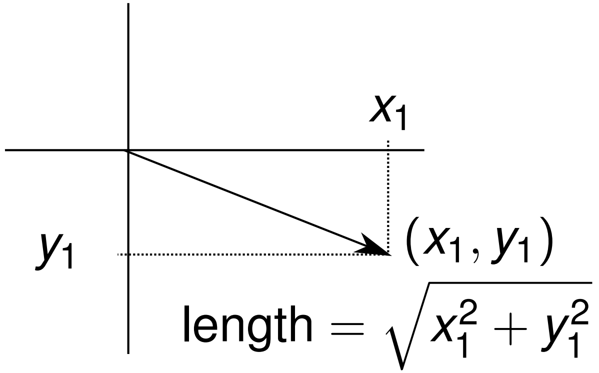


Figure: Length of vector $\langle x_1, y_1 \rangle$.

Examples

Example: Let $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$. Find $|\mathbf{u}|$ and $|\mathbf{v}|$.

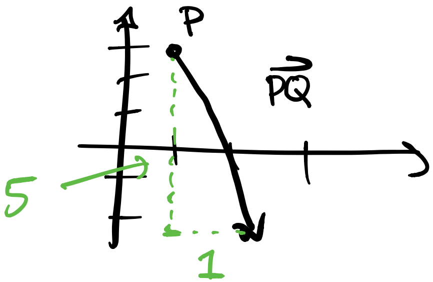
$$|u| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$|v| = \sqrt{(-1)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

u, v have different directions but the same length.

Example: Let $\mathbf{u} = \langle 1, 2\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$. Find $|\mathbf{u}|$ and $|\mathbf{v}|$.

Example: Let $P = (1, 3)$ and $Q = (2, -2)$. Compute $|\overrightarrow{PQ}|$.



$$|\overrightarrow{PQ}| = \sqrt{1^2+5^2} = \sqrt{1+25} = \sqrt{26}$$

Also note that $\overrightarrow{PQ} = \langle 2-1, -2-3\rangle$

$$= \langle 1, -5\rangle$$

so, $|\overrightarrow{PQ}| = \sqrt{1^2+(-5)^2} = \sqrt{26}$.

Example: Let $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$. Find $|\mathbf{u}|$ and $|\mathbf{v}|$.

Example: Let $P = (1, 3)$ and $Q = (2, -2)$. Compute $|\overrightarrow{PQ}|$.

Exercise: Let $\mathbf{u} = \langle 3, 6 \rangle$. Compute $|\mathbf{u}|$. How is $\mathbf{u}$ related to $\langle 1, 2 \rangle$?

$|\vec{u}| = |\langle 3,6 \rangle| = \sqrt{3^2+6^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$

$\vec{u} = 3\langle 1,2 \rangle$ so they are parallel vectors.

Notice that $|\langle 1,2 \rangle| = \sqrt{5}$ from before and

$|\vec{u}| = 3\sqrt{5} = 3|\langle 1,2 \rangle|$.

$\vec{u} = 3\langle 1,2 \rangle$

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Exercise: Let $\mathbf{u} = \langle 3, 6 \rangle$. Compute $|\mathbf{u}|$. How is $\mathbf{u}$ related to $\langle 1, 2 \rangle$?

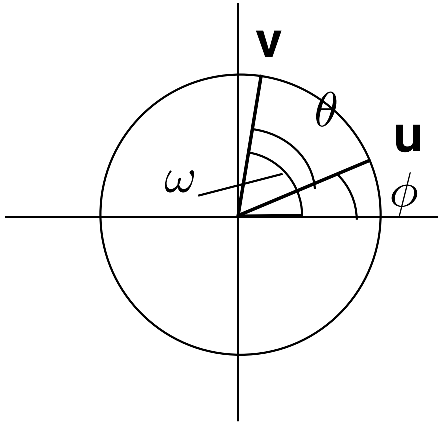
Exercise: Let $P = (2, 1)$ and $Q = (6, -2)$. Compute $|\overrightarrow{PQ}|$.

$$\overrightarrow{PQ} = \langle 6-2, -2-1 \rangle = \langle 4, -3 \rangle$$

so, $|\overrightarrow{PQ}| = |\langle 4, -3 \rangle| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5.$

Unit vectors

Directions are given by an *angle* which can be seen on the unit circle.



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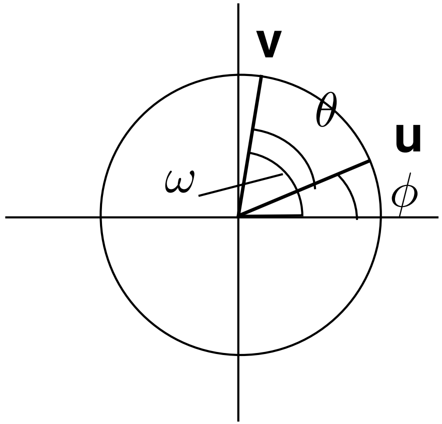

Figure: Unit vector.

Position vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ has

$$|\mathbf{u}| = (\cos^2\theta + \sin^2\theta)^{1/2} = 1.$$

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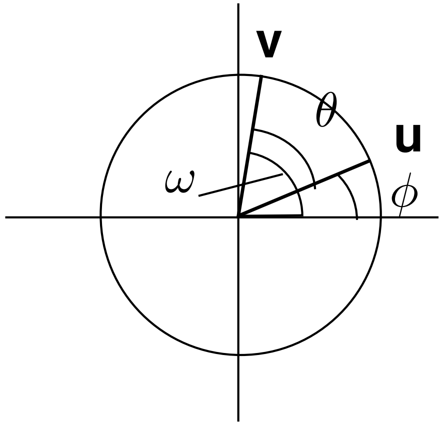

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Any vector $\mathbf{u}$ with $|\mathbf{u}| = 1$ is called a *unit vector*. Think of them as directions on the unit circle.

Vectors to unit vectors

Let **v** be a vector with $\mathbf{v} \neq \mathbf{0}$. Then $|\mathbf{v}| \neq 0$ (why?) Now, $|\mathbf{v}|$ is a *number* so scalar.

Remark: In this class it's very important to remember what is a scalar vs. what is a vector. You can multiply scalars together and multiply vectors by scalars, but you cannot multiply two vectors.

$\vec{v} = \vec{0}$ is the only vector with $|\vec{v}| = 0$.

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Let $\mathbf{v}$ be a vector with $\mathbf{v} \neq \mathbf{0}$. Then $|\mathbf{v}| \neq 0$ (why?) Now, $|\mathbf{v}|$ is a *number* so scalar.
Let $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$. $\mathbf{u}$ is a positive scalar multiple of $\mathbf{v}$ so it points in same direction.

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Let $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$. **u** is a positive scalar multiple of **v** so it points in same direction.
Compute: if $\mathbf{v} = \langle x_1, y_1 \rangle$, $|\mathbf{v}| = (x_1^2 + y_1^2)^{1/2}$, so

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \left\langle \frac{x_1}{(x_1^2 + y_1^2)^{1/2}}, \frac{y_1}{(x_1^2 + y_1^2)^{1/2}} \right\rangle$$

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and

$$|\mathbf{u}| = \left(\frac{x_1^2}{(x_1^2 + y_1^2)} + \frac{y_1^2}{(x_1^2 + y_1^2)} \right)^{1/2} = 1$$

Examples

Observe that if $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$, $\mathbf{v} = |\mathbf{v}|\mathbf{u}$. This is nice because it is easy to see *length* $= |\mathbf{v}|$ and *direction* $= \mathbf{u}$ *unit vector*.

Example: Let $\mathbf{v} = \langle 3, 4 \rangle$. Find a unit vector in the direction of $\mathbf{v}$.

$|\vec{v}| = |\langle 3,4\rangle| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Let $\vec{u}$ denote the unit vector. Then

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}\langle 3,4\rangle = \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle.$$

Examples

Exercise: Let $\mathbf{v} = \langle 1, -\sqrt{3} \rangle$. Find a unit vector in the direction of $\mathbf{v}$.

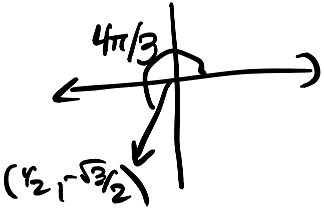
$$|\vec{v}| = |\langle 1, -\sqrt{3} \rangle| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

Let $\vec{u}$ denote the unit vector. Then

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{2}\langle 1, -\sqrt{3} \rangle = \langle 1/2, -\sqrt{3}/2 \rangle$$

Bonus Question: What is the direction in radians?

$\frac{4\pi}{3}$



Notation

Unit vectors in x and y directions are important enough to get their own notation.

$$\mathbf{i} = \langle 1, 0 \rangle \,, \;\; \mathbf{j} = \langle 0, 1 \rangle \,.$$

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Unit vectors in x and y directions are important enough to get their own notation.

$$\mathbf{i} = \langle 1, 0 \rangle, \ \mathbf{j} = \langle 0, 1 \rangle.$$

Note:

$$\langle x_1, y_1 \rangle = x_1 \mathbf{i} + y_1 \mathbf{j}$$

says "go x_1 in the direction of **i** (x direction), then go y_1 in the direction of **j**".

We have just covered vectors in 2-dimensions, but we can extend the idea to 3-dimension and more.

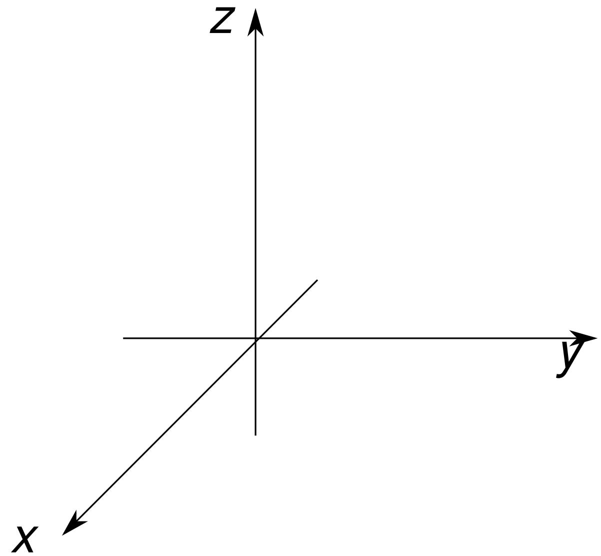
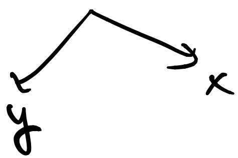
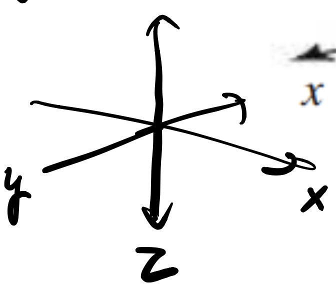


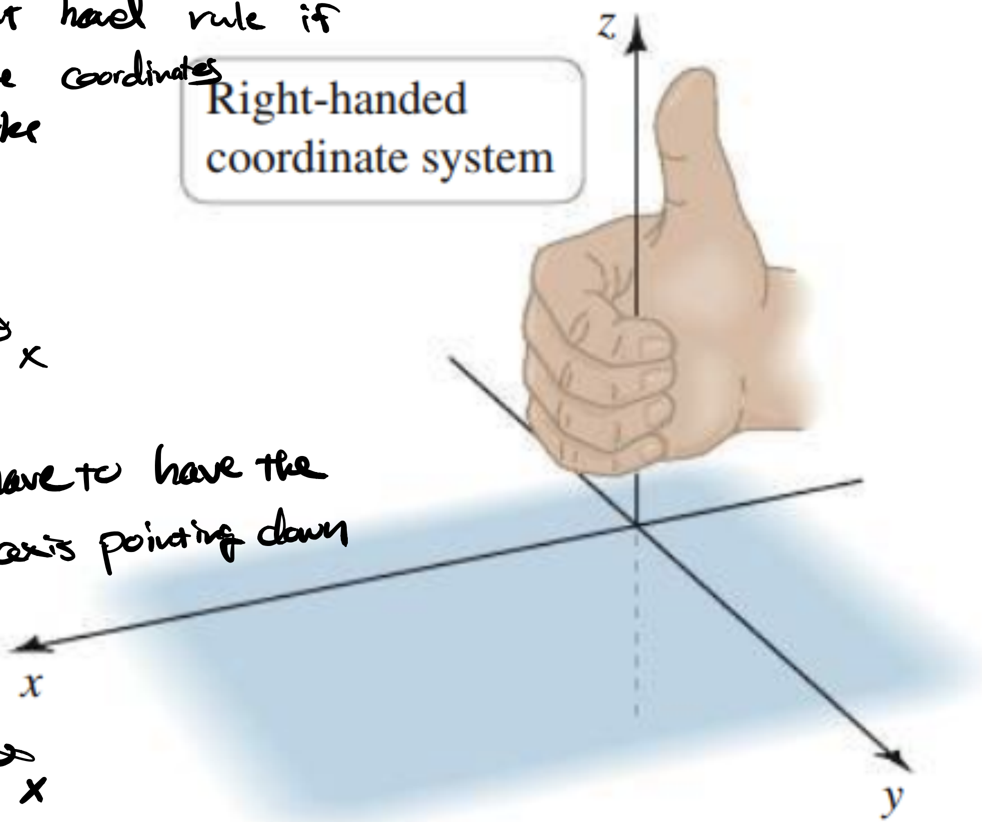
Figure: $\mathbb{R}^3$ and right hand rule.

By the right hand rule if we wanted the coordinates to look like



Then we have to have the positive z axis pointing down





Points/vectors in $\mathbb{R}^3$

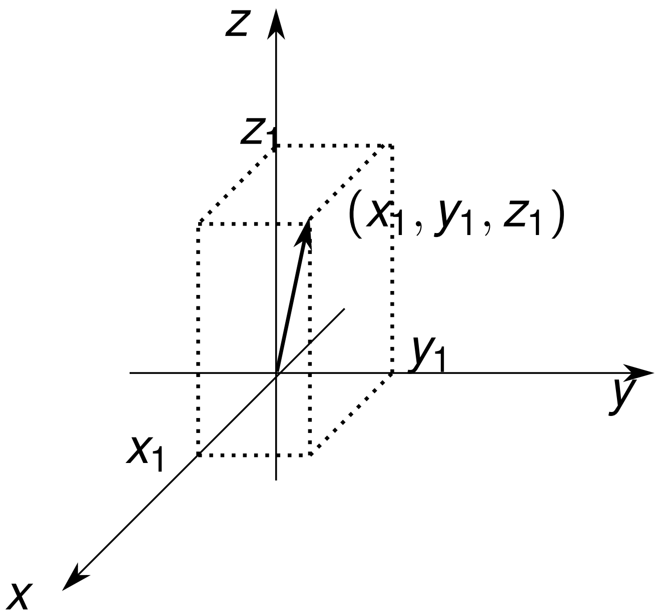


Figure: $\mathbb{R}^3$ and a point (x_1, y_1, z_1).

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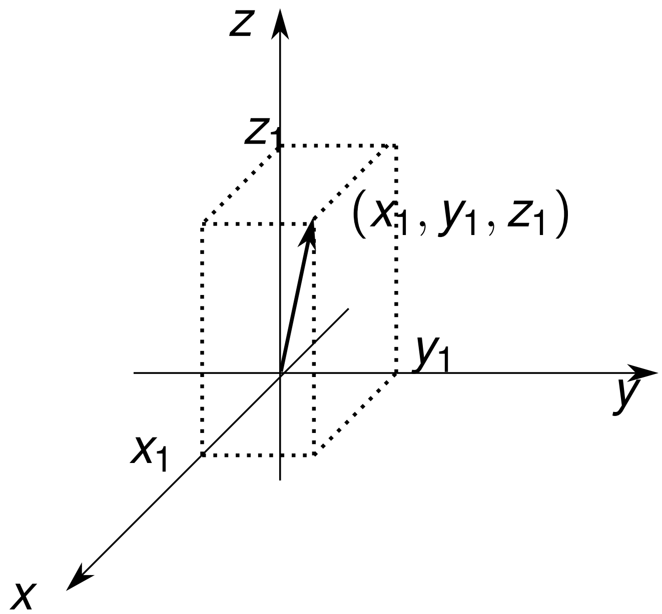
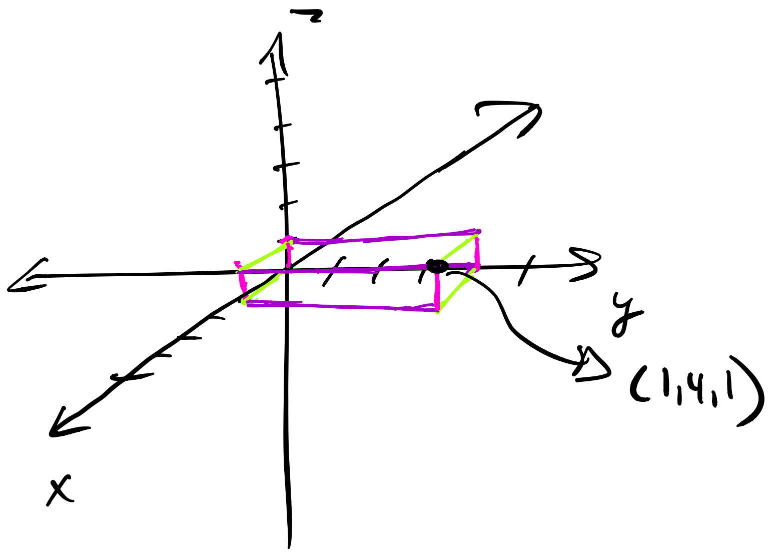


Figure: $\mathbb{R}^3$ and a point (x_1, y_1, z_1).

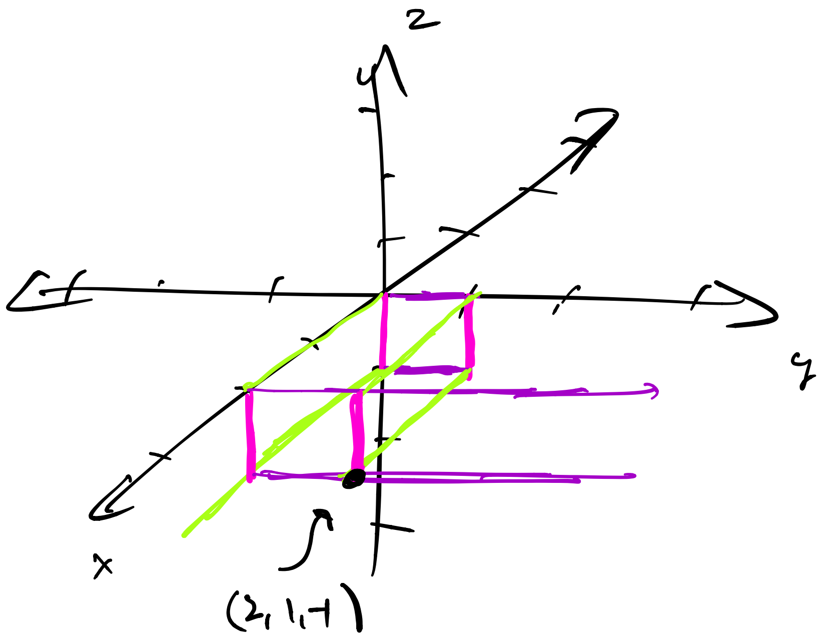
Position vector $\mathbf{u} = \langle x_1, y_1, z_1 \rangle$ says "go x_1 in the x direction, then y_1 in the y direction, then z_1 in the z direction."

Example: Plot the point $(1, 4, 1)$.



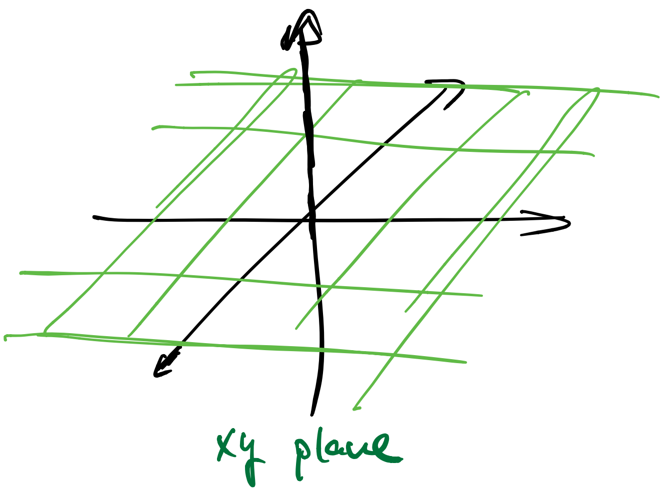
Example: Plot the point $(1, 4, 1)$.

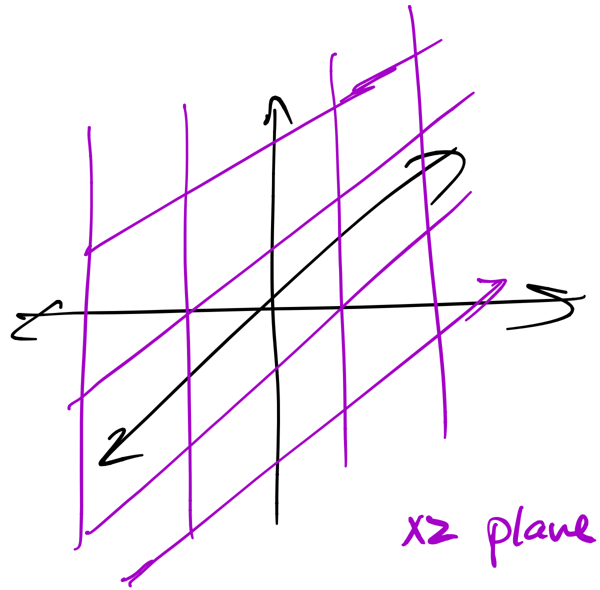
Exercise: Plot the point $(2, 1, -1)$.



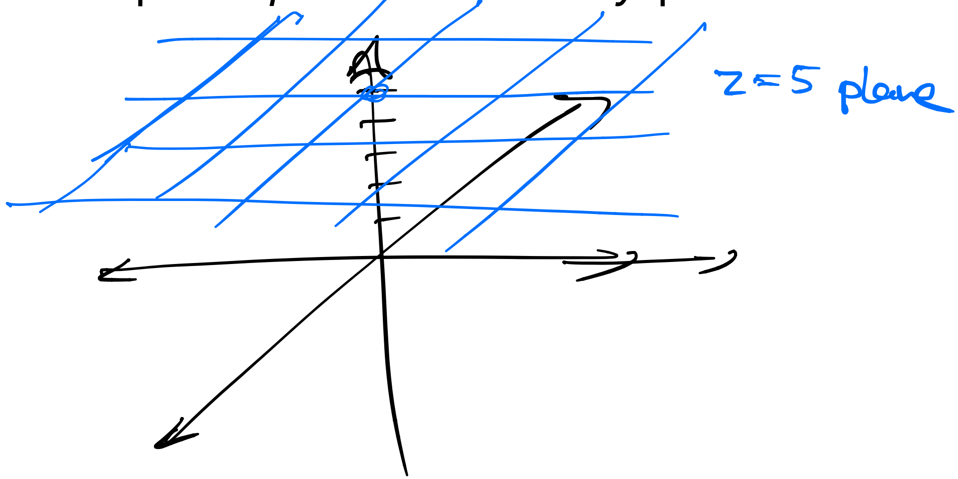
Planes

When $z = 0$, called xy plane, $y = 0$ the xz plane, and $x = 0$ the yz plane.





When $z = 0$, called xy plane, $y = 0$ the xz plane, and $x = 0$ the yz plane.
If $z = 5$, this is still a plane *parallel to* the xy plane.



Planes

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If $z = 5$, this is still a plane *parallel to* the xy plane.
Note that the sides of the box live in planes parallel to the xy, xz, and yz planes.

Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are similar to vectors in $\mathbb{R}^2$. They have a length and direction. Or point from a point $P = (x_1, y_1, z_1)$ to $Q = (x_2, y_2, z_2)$ ($\overrightarrow{PQ}$).

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$$\overrightarrow{PQ} = \langle x_2, y_2, z_2 \rangle - \langle x_1, y_1, z_1 \rangle .$$

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Example: $P = (1, 2, -1)$, $Q = (0, 4, 2)$.

$$\overrightarrow{PQ} = \langle 0 - 1, 4 - 2, 2 - (-1) \rangle = \langle -1, 2, 3 \rangle .$$

Vector operations in $\mathbb{R}^3$

Vectors add and scalar multiply:

$$\langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle ,$$

and for scalar a,

$$a \langle x_1, y_1, z_1 \rangle = \langle ax_1, ay_1, az_1 \rangle .$$

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Example: Let $\mathbf{u} = \langle 2, 1, 0\rangle$ and $\mathbf{v} = \langle -1, -1, 1\rangle$. Compute $\mathbf{u} + \mathbf{v}$ and $2\mathbf{u} - 4\mathbf{v}$.

$\vec{u}+\vec{v} = \langle 2,1,0\rangle + \langle -1,-1,1\rangle = \langle 2+-1, 1+-1, 0+1\rangle = \langle 1,0,1\rangle$

$2\vec{u} - 4\vec{v} = 2\langle 2,1,0\rangle + -4\langle -1,-1,1\rangle = \langle 4,2,0\rangle + \langle 4,4,-4\rangle$

$= \langle 8,6,-4\rangle$

Length in $\mathbb{R}^3$ is a little more complicated. Look at box:

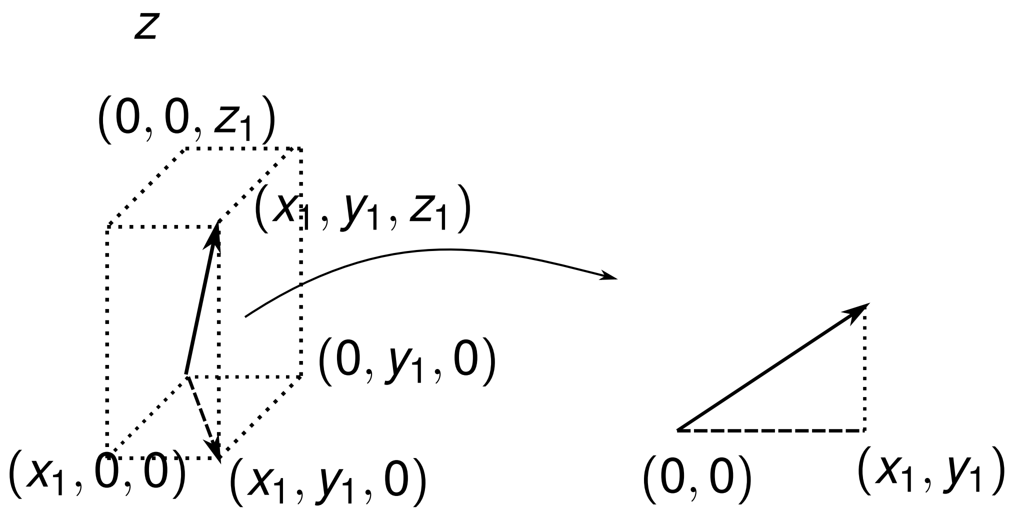

Figure: The triangle with bottom edge in xy plane.

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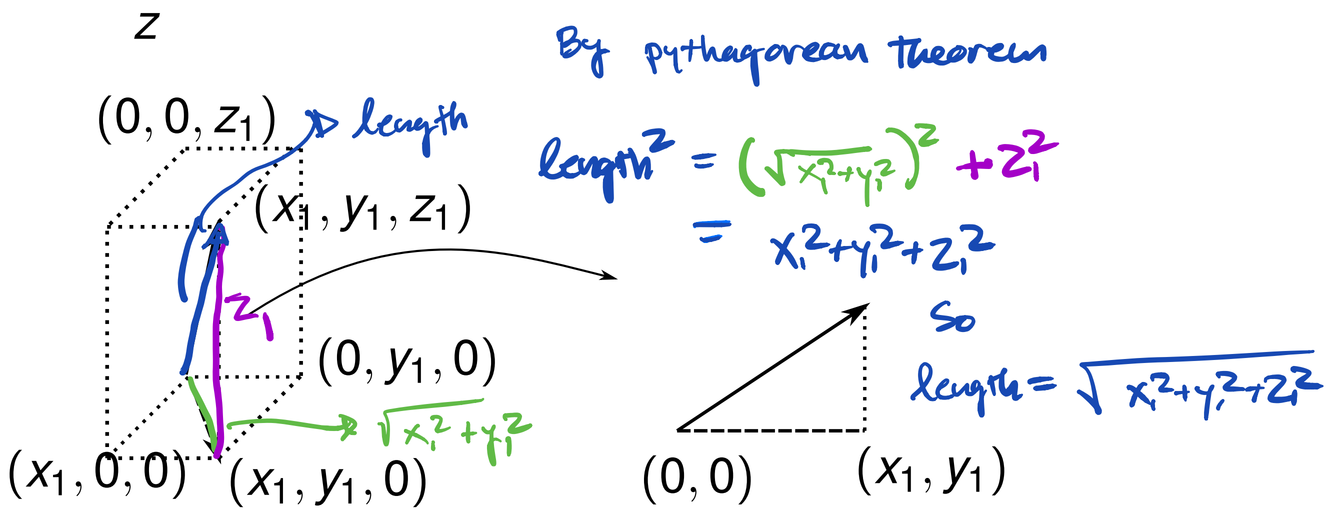

Figure: The triangle with bottom edge in *xy* plane.

Bottom has length $(x_1^2 + y_1^2)^{1/2}$ so diagonal has length $(x_1^2 + y_1^2 + z_1^2)^{1/2}$.

Example

Example: Let $\mathbf{u} = \langle 2, 1, 0 \rangle$. Compute $|\mathbf{u}|$.

$$|\vec{u}| = |\langle 2,1,0\rangle| = \sqrt{2^2+1^2+0^2} = \sqrt{4+1+0} = \sqrt{5}$$

Example

Example: Let $\mathbf{u} = \langle 2, 1, 0 \rangle$. Compute $|\mathbf{u}|$.
Exercise: Let $\mathbf{v} = \langle -1, -1, 1 \rangle$. Compute $|\mathbf{v}|$.

$$|\vec{v}| = |\langle -1, -1, 1 \rangle| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}$$

Application: spheres

In $\mathbb{R}^2$ the unit circle is the collection of all unit vectors with tail at $(0,0)$: If $\mathbf{u} = \langle x, y \rangle$, then it is on the unit circle if $|\mathbf{u}| = (x^2 + y^2)^{1/2} = 1$.

Application: spheres

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$$(x^2 + y^2 + z^2)^{1/2} = r \rightsquigarrow x^2 + y^2 + z^2 = r^2.$$

A sphere of radius r centered at (a, b, c) is all vectors of length r with tail at (a, b, c). $\mathbf{u} = \langle x - a, y - b, z - c \rangle$, $\mathbf{u}$ points from (a, b, c) to (x, y, z), so sphere is

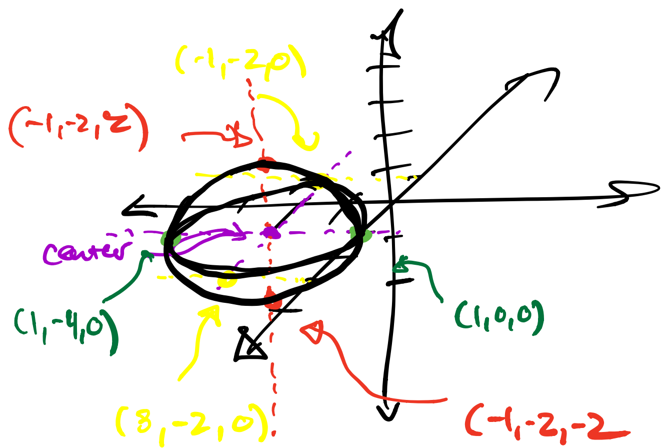
$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Example: Find an equation for the sphere of radius 2 centered at the point $(1, -2, 0)$. Sketch.

Sphere of radius 2 is $x^2+y^2+z^2=2^2$ but we want it with center $(1,-2,0)$ so we need to shift it. this gives

$$(x-1)^2+(y-(-2))^2+(z-0)^2=4$$

$$(x-1)^2+(y+2)^2+z^2=4$$



Example: Find an equation for the sphere of radius 2 centered at the point $(1, -2, 0)$. Sketch.

Exercise: What object is described by the equation

$$x^2 + y^2 + z^2 - 2x + 2z = 0.$$

Sphere

We get it into the right form by completing the square.

$$x^2 - 2x + y^2 + z^2 + 2z = 0$$

$$(x^2 - 2x + 1) + y^2 + (z^2 + 2z + 1) = 1 + 1$$

$$(x-1)^2 + y^2 + (z+1)^2 = 2$$

Properties of vector operations

Lets consider $\vec{u} = \langle x_1, y_1 \rangle$, $\vec{v} = \langle x_2, y_2 \rangle$ $\vec{w} = \langle x_3, y_3 \rangle$

- Addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$\vec{u} + \vec{v} = \langle x_1 + x_2, y_1 + y_2 \rangle = \langle x_2 + x_1, y_2 + y_1 \rangle = \vec{v} + \vec{u}$ (commutivity of addition)

Note: The 3D and higher dimensions are similar.

Properties of vector operations

- Addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Addition is associative:

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

By associativity of numbers

$$(\vec{u}+\vec{v})+\vec{w} = \langle (x_1+x_2)+x_3, (y_1+y_2)+y_3 \rangle = \langle x_1+(x_2+x_3), y_1+(y_2+y_3) \rangle$$

$$= \vec{u}+(\vec{v}+\vec{w})$$

Same idea holds for higher dimensions

Properties of vector operations

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- Addition is associative:

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

- $\mathbf{0} = \langle 0, 0 \rangle$ is additive identity: for any vector $\mathbf{u}$,

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

$$\vec{u} + \vec{0} = \langle x_1 + 0, y_1 + 0 \rangle = \langle x_1, y_1 \rangle = \vec{u}$$

Properties of vector operations

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- Additive inverse:

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

$-\vec{u} = \langle -x_1, -y_1 \rangle$

$\vec{u} + (-\vec{u}) = \langle x_1 + (-x_1), y_1 + (-y_1) \rangle = \vec{0}$

same idea holds for higher dimensions

Properties of vector operations

- Addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
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- Additive inverse:

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

- Scalar multiplication distributes: if a and b are real numbers,

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

$a(\vec{u}+\vec{v}) = \langle a(x_1+x_2), a(y_1+y_2)\rangle = \langle ax_1+ax_2, ay_1+ay_2\rangle = a\vec{u}+a\vec{v}$

and

$$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$$

same holds in higher dimensions

$(a+b)\vec{u} = \langle (a+b)x_1, (a+b)y_1\rangle$
$= \langle ax_1+bx_1, ay_1+by_1\rangle = a\vec{u}+b\vec{u}$

More properties

- Multiplication by scalar 0: $0\mathbf{u} = \mathbf{0}$

$0\vec{u} = \langle 0 \cdot x_1, 0 \cdot y_1 \rangle = \langle 0, 0 \rangle = \vec{0}$

More properties

- Multiplication by scalar 0: $0\mathbf{u} = \mathbf{0}$
- Multiplication of $\mathbf{0}$: if c is a real number, $c\mathbf{0} = \mathbf{0}$

$c\vec{0} = \langle c\cdot 0, c\cdot 0\rangle = \langle 0, 0\rangle = \vec{0}$

More properties

- Multiplication by scalar 0: $0\mathbf{u} = \mathbf{0}$
- Multiplication of $\mathbf{0}$: if c is a real number, $c\mathbf{0} = \mathbf{0}$
- Multiplicative identity: $1\mathbf{u} = \mathbf{u}$

$1\vec{u} = \langle 1\cdot x_1, 1\cdot y_1 \rangle = \langle x_1, y_1 \rangle = \vec{u}$

More properties

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- Scalar multiplication is commutative: if a, b are real numbers

$$ab\mathbf{u} = ba\mathbf{u}$$

$$ab\vec{u} = \langle abx_1, aby_1\rangle = \langle bax_1, bay_1\rangle = ba\vec{u}$$

(→ commutativity of multiplication)

More properties

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$$(ab)\mathbf{u} = a(b\mathbf{u})$$

$(ab)\vec{u} = \langle (ab)x_1, (ab)y_1 \rangle = \langle a(bx_1), a(by_1) \rangle = a(b\vec{u})$ (associativity of multiplication)

More properties

- Multiplication by scalar 0: $0\mathbf{u} = \mathbf{0}$
- Multiplication of $\mathbf{0}$: if c is a real number, $c\mathbf{0} = \mathbf{0}$
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- Scalar multiplication is associative: if a, b are real numbers

$$(ab)\mathbf{u} = a(b\mathbf{u})$$

- Length scales: if a is a real number, then

$$|a\mathbf{u}| = |a||\mathbf{u}|$$

(Why $|a|$?) $|a\vec{u}| = \sqrt{a^2x_1^2 + a^2y_1^2} = \sqrt{a^2(x_1^2+y_1^2)} = \sqrt{a^2}\sqrt{x_1^2+y_1^2} = |a|\,|\vec{u}|$

Vectors in $\mathbb{R}^n$

We can extend the idea of vectors to n-dimensions. A vector **u** would have the form $\mathbf{u} = \langle x_1, x_2, x_3, \cdots, x_n \rangle$. If $\mathbf{v} = \langle y_1, y_2, y_3, \cdots, y_n \rangle$ is another vector in $\mathbb{R}^n$ then

$$\mathbf{u} + \mathbf{v} = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots, x_4 + y_4 \rangle$$

Multiplication by a scalar *a* is defined by

$$a\mathbf{u} = \langle ax_1, ax_2, ax_3, \cdots, ax_n \rangle$$

We can also define magnitude or length in higher dimensions by

$$|\mathbf{u}| = (x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)^{1/2}$$

In higher dimensions we still have all of the properties listed on the last few slides.