Tangent plane and linear approximation

July 5, 2020

Big Picture for today

Big Picture:
The gradient can be used to find the *tangent plane*. As an application, we get linear approximation of functions.

Gradient ⊥ to level sets

In recitation
↗ and I talked about
this briefly on Thursday

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Gradient $\perp$ to level sets

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Theorem: Let D be a domain in $\mathbb{R}^2$, $f : D \to \mathbb{R}$ a differentiable function. Let (a, b) be a point in D and let $z_0 = f(a, b)$. Then if $\nabla f(a, b) \neq \mathbf{0}$ and the level set $f(x, y) = z_0$ is smooth at (a, b), then $\nabla f(a, b)$ is perpendicular to the level set $f(x, y) = z_0$.

Tangent lines in $\mathbb{R}$

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Continued

Remember level sets are circles

Rephrase: Let $F(x, y) = x^2 + y^2 - 1$ so curve $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ lives on the *level set* where $F = 0$.

Continued

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Same idea works for tangent *planes* to a surface in $\mathbb{R}^3$. If $F(x, y, z)$ is a differentiable function on $\mathbb{R}^3$, and the level set $F(x, y, z) = 0$ (or any constant) is smooth, it describes a *surface*.

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Rephrase: Let $F(x, y) = x^2 + y^2 - 1$ so curve $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ lives on the *level set* where $F = 0$. We know that ∇F is perpendicular to level set $\rightsquigarrow$ perpendicular to curve $\mathbf{r}(t)$ $\rightsquigarrow$ tangent is any vector perpendicular to ∇F.
Same idea works for tangent *planes* to a surface in $\mathbb{R}^3$. If $F(x, y, z)$ is a differentiable function on $\mathbb{R}^3$, and the level set $F(x, y, z) = 0$ (or any constant) is smooth, it describes a *surface*. Then ∇F is *normal* to this surface $\rightsquigarrow$ normal to tangent plane. $\rightsquigarrow$ If (a, b, c) is in the level set $F(a, b, c) = 0$, tangent plane is given by $\mathbf{n} \cdot \langle x - a, y - b, c - d \rangle = 0$, or equivalently

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, c - d \rangle = 0.$$

Example

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Example: Consider surface $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$.

Example

This is a sphere of radius 1.

Example: Consider surface $\overbrace{x^2+y^2+z^2=1}$ in $\mathbb{R}^3$. If $F(x,y,z)=x^2+y^2+z^2-1$, surface is where $F=0$.

Example

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$$\nabla F = \langle 2x, 2y, 2z \rangle \rightsquigarrow \nabla F(0, 0, 1) = \langle 0, 0, 2 \rangle .$$

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This is our normal vector, so tangent plane is given by

$$\langle 0, 0, 2 \rangle \cdot \langle x - 0, y - 0, z - 1 \rangle = 0,$$

or $z = 1$.

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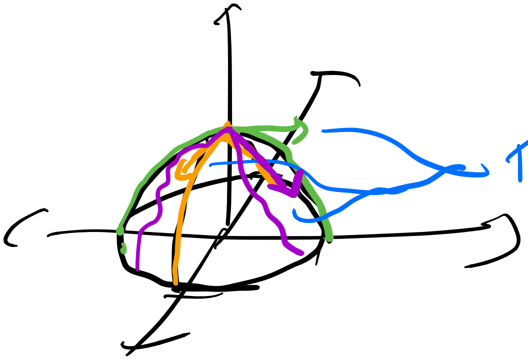
or $z = 1$. Parallel to xy plane going through $(0, 0, 1)$.

Alternative View of Tangent Plane

Consider some surface $F(x, y) = z$. Let $\mathbf{r}(t) =< x(t), y(t), F(x(t), y(t)) >$ be a curve on the surface $z = F(x, y)$. Then $\mathbf{r}'(t_0)$ is tangent to the surface at $< x(t_0), y(t_0), F(x(t_0), y(t_0)) >$. Using this idea we say that the tangent plane for the surface at $(x(t_0), y(t_0), F(x(t_0), y(t_0)))$ is composed of all vectors $\mathbf{r}'(t_0)$ such that there is a differentiable curve $\mathbf{r}(t)$ with $\mathbf{r}(t_0) =< x(t_0), y(t_0), F(x(t_0), y(t_0)) >$.



Three possible tangent vectors

Also, we could rescale these curves to get larger tangent vectors

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Two examples would be the curves
$< x(t_0) + t, y(t_0), F(x(t_0) + t, y(t_0)) >$ and
$< t_0, y(t_0) + t, F(x(t_0), y(t_0) + t)) >$ which would go through the points at $t = 0$ in both situations. Furthermore we would get that $\mathbf{r}'(0) =< 1, 0, \partial_x F(x(t_0), y(t_0)) >$ and
$\mathbf{r}'(0) =< 0, 1, \partial_y F(x(t_0), y(t_0)) >$.

Alternative View of Tangent Plane

Let's compare this to the last example. We were looking at the surface $x^2 + y^2 + z^2 = 1$ at the point $(0, 0, 1)$. We can rewrite this as $F(x, y) = z = \sqrt{1 - x^2 - y^2}$ since we are in the positive portion. Then, $F_x = \frac{-x}{\sqrt{1-x^2-y^2}}$ and $F_y = \frac{-y}{\sqrt{1-x^2-y^2}}$. Along the curves from the last slides we get that tangent vectors are $< 1, 0, 0 >$ and $< 0, 1, 0 >$ respectivly. This means that normal vector is given by $< 1, 0, 0 > \times < 0, 1, 0 >= \mathbf{k}$ which agrees with the direction of the gradient.

Example

Consider surface $x^2 + 4y^2 - z = 4$. Find tangent plane at the point $(2, 0, 0)$.

$$F(x,y,z) = 4$$

$$\nabla F = \langle 2x, 8y, -1 \rangle$$

$$\nabla F(2,0,0) = \langle 4, 0, -1 \rangle$$

Plane $\nabla F(2,0,0) \cdot \langle x-2, y, z \rangle$

$$= \langle 4, 0, -1 \rangle \cdot \langle x-2, y, z \rangle$$

$$= 4x - 8 - z = 0 \rightarrow \boxed{4x - z = 8}$$

Example

Consider surface $x^2 + 4y^2 - z = 4$. Find tangent plane at the point $(2, 0, 0)$. If $F(x, y, z) = x^2 + 4y^2 - z - 4$, then surface is $F(x, y, z) = 0$.

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or

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Note: If $f(x, y) = x^2 + 4y^2 - 4$, then the *graph* $z = x^2 + 4y^2 - 4$ is the same as the level set $F(x, y, z) = 0$.

Exercise

Exercise: Find the tangent plane to the surface $z = x^2y^3$ at the point $(1, 2, 8)$.

$$F(x,y,z) = x^2y^3 - z$$

$$\nabla F = \langle 2xy^3,\ 3x^2y^2,\ -1\rangle$$

$$\nabla F(1,2,8) = \langle 16, 12, -1\rangle$$

Plane:

$$0 = \nabla F \cdot \langle x-1, y-2, z-8\rangle = \langle 16, 12, -1\rangle \cdot \langle x-1, y-2, z-8\rangle$$

$$= 16x - 16 + 12y - 24 - z + 8$$

$$\Rightarrow 16x + 12y - z = 32$$

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Plane is

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$$g(x) = \ell(x) + r$$

where $r =$ remainder, $r \to 0$ faster than $(x - c)$.

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where $r =$ remainder, $r \to 0$ faster than $(x - c)$.
This is *Taylor's formula* with remainder.

What this means is that

$$\lim_{x \to c} \frac{r(x)}{(x-c)} = 0$$

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$$\lim_{(x,y)\to(a,b)} \frac{R(x,y)}{((x-a)^2+(y-b)^2)^{1/2}} = 0.$$

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where $R \to 0$ faster than $(x-a)$ and $(y-b)$ at the same time. (That is as $((x-a)^2 + (y-b)^2)^{1/2} \to 0$).
Note: This is really just the tangent plane to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$.

↳ why? This becomes $0 = f(x,y) - z = F(x,y,z)$

$\Rightarrow \nabla F = \langle \partial_x f, \partial_y f, -1 \rangle$

$\Rightarrow$ plane is $0 = \nabla F \cdot \langle x-a, y-b, z-f(a,b) \rangle$

$$0 = \partial_x f\,(x-a) + \partial_y f\,(y-b) - z + f(a,b)$$

Remember $z = f(x,y) \Rightarrow$

$$f(x,y) = \partial_x f(x-a) + \partial_y f(y-b) + f(a,b)$$

Why does tangent plane approximation matter?

- Linear functions are easier to handle
- Computers have to approximate irrational numbers by linear/polynomial functions.

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so

$$\begin{aligned} L(x, y) &= f(-2, 1) + \nabla f(-2, 1) \cdot \langle x + 2, y - 1 \rangle \\ &= 4e^2 - 6 + (-4e^2 + 3)(x + 2) + 8e^2(y - 1) \end{aligned}$$

is best linear approximation.

Remark

We just talked about linear approximations but you can extend this further. There is an extension of Taylor's Theorem to higher dimensions. You can approximate differentiable functions by polynomials in multiple variables. This topic will not be covered in our course, but there are plenty of resources online to learn about it for those interested.

let $f(x,y) = xy + x - y$; using the point $(2,3)$ estimate $f(2.05, 3.05)$

$\nabla f = \langle y+1,\ x-1 \rangle$ $\quad f(2,3) = 6+2-3 = 5$

$\nabla f(2,3) = \langle 4,1 \rangle$

$f(x,y) \approx 5 + 4(x-2) + 1(y-3)$

$f(2.05, 3.05) \approx 5 + 4 \cdot .05 + 1 \cdot 0.5 = \boxed{5.25}$

Real Value $f(2.05, 3.05) = 5.2525$ Difference is 0.0025

Let $f(x,y) = xy + x - y$; Using the point $(2,3)$ estimate $f(3,4)$.

From above, $f(x,y) \approx 5 + 4(x-2) + (y-3)$

$f(3,4) \approx 5 + 4 + 1 = 10$

Real Value

$f(3,4) = 12 + 3 - 4 = 11$

Difference is 1

notice how the difference grows the further away you are. (Think Taylor's remainder Theorem)

Taylor's Remainder Theorem

Definition: (Multi-indices 2D)

Let $\alpha = (\alpha_1, \alpha_2)$ Then

$$D^\alpha f = \partial_x^{\alpha_1} \partial_y^{\alpha_2} f, \quad \alpha! = \alpha_1!\,\alpha_2!$$

and $|\alpha| = \alpha_1 + \alpha_2$

Taylor's Remainder Theorem (2D)

Suppose $f \in C^\infty$ (i.e you can differentiate as many times as you want). Then,

$$f(x,y) = f(a,b) + \overbrace{\partial_x f(a,b)(x-a) + \partial_y f(a,b)(y-b)}^{\text{Linear approx}}$$

$$\text{second order approx.}\left\{\begin{array}{l} + \frac{1}{2}\partial_x^2 f(a,b)\,(x-a)^2 + \frac{1}{2}\partial_x \partial_y f(a,b)\,(x-a)(y-b) \\ + \frac{1}{2}\partial_y \partial_x f(a,b)\,(y-b)(x-a) + \frac{1}{2}\partial_y^2 f(a,b)\,(y-b)^2 \end{array}\right.$$

$$+ \sum_{|\alpha|=3}^{\infty} \frac{D^\alpha f(a,b)}{\alpha!}(x-a)^{\alpha_1}(y-b)^{\alpha_2}$$

Higher Dimensions and Full Theorem

Definition: (Multi-indices $\mathbb{R}^n$) → Variables are $(x_1, \dots, x_n)$

$\alpha = (\alpha_1, \dots, \alpha_n)$

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$

$\alpha! = \alpha_1!\,\alpha_2! \cdots \alpha_n!$

$$D^\alpha f = \partial_{x_1}^{\alpha_1} f\, \partial_{x_2}^{\alpha_2} f \dots \partial_{x_n}^{\alpha_n} f$$

$$(\vec{x} - \vec{a})^\alpha = (x_1 - a_1)^{\alpha_1}(x_2 - a_2)^{\alpha_2} \dots (x_n - a_n)$$

Taylor's Theorem: Let $f:\mathbb{R}^n \to \mathbb{R}$ be a $(k+1)$ differentiable function on $\mathbb{R}$. Let $\vec{a} \in \mathbb{R}^n$ be a point. Then,

$$f(\vec{x}) = \sum_{|\alpha|=0}^{k} \frac{D^\alpha f(\vec{a})}{\alpha!}(\vec{x}-\vec{a})^\alpha + \sum_{|\alpha|=k+1} R_\alpha(\vec{x})(\vec{x}-\vec{a})^\alpha$$

where
$$|R_\alpha(\vec{x})| \leq \frac{1}{\alpha!} \max_{|\beta|=|\alpha|} \max_{|\vec{y}-\vec{a}| \leq |\vec{a}-\vec{x}|} |D^\beta f(\vec{y})|$$

Let $|\vec{x}-\vec{a}| = r$. Then this says we can approximate a differentiable function by a k-th order polynomial up to a remainder which is a $(k+1)$ order polynomial and which is bounded by $r^{k+1} \times$ maximum value of any $k+1$ order derivative on the ball of radius r around $\vec{a}$.

Remark: If r is small this gives us a really close approximation.