Local Min/max

July 5, 2020

Big Picture for today

Big Picture: Finding *local* min or max is similar to in $\mathbb{R}$: Find critical points and use a second derivative test if possible.

Local min/max in $\mathbb{R}$

Recall: If $g(x)$ is a function on an interval (c, d), then g has a *local minimum* (or maximum) at x_0 in (c, d) if there is an open interval (c', d') in (c, d) such that $g(x) \geq g(x_0)$ (or $g(x) \leq g(x_0)$) on (c', d').

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6) Converse *not true*: $g(x) = x^3 + 1$

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Notation: If $\nabla f(a, b) = \mathbf{0}$, then (a, b) is a *critical point*.

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$\nabla f = \langle y, x \rangle$, so only critical point is $(0, 0)$. Level sets $\rightsquigarrow$ neither max nor min. Called a *saddle point*.

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There are two possibilities from the first equation and two from the second equation, so there would be $2 \times 2 = 4$ possibilities.

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First equation, first condition: $y = 0$. Plug in to second equation: $x(0 - 2) = 0$ only if $x = 0$. That gives the point $(0, 0)$.

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First equation, second condition: $y = 2$.

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Note: The second condition from the second equation $y = 1$ is not compatible with the first equation, so doesn't show up as a critical point.

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Plug into second equation $\rightsquigarrow x^2 + 0 - 4 = 0$, so $x = \pm 2$. $\rightsquigarrow$
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Plug into second equation $\rightsquigarrow$ $x^2 + 0 - 4 = 0$, so $x = \pm 2$. $\rightsquigarrow$ $(2,0)$ and $(-2,0)$ are critical points.
If $y = -2x$, $x^2 + 2x(-2x) - 4 = -3x^2 - 4 \neq 0$, so there are not critical points with $y = -2x$.

Second derivative test

Recall: Suppose $g(x)$ has two continuous derivatives and has a critical points at x_0. Then if $g''(x_0) > 0$ it's a min and if $g''(x_0) < 0$ it's a max.

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Assume all second order partials are continuous, and let $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$.
Theorem: If (a, b) is a critical point for f and

- $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local max
- $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local min
- $D(a, b) < 0$ then (a, b) is a saddle point
- $D(a, b) = 0$ no information

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$\nabla f = \langle y, x \rangle = \mathbf{0}$ when $x = 0$ and $y = 0$. $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$, and $f_{xy}(0, 0) = 1$, so $D = 0 - 1 < 0$, so it is a saddle point.

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$\nabla f = \langle y(y - 2), x(2y - 2) \rangle$. $f_{xx} = 0$, $f_{yy} = 2x$, $f_{xy} = 2y - 2$. At $(0, 0)$, $D(0, 0) = -(-2)^2 < 0$ and at $(0, 2)$, $D(0, 2) = -(2)^2 < 0$, so both saddle points.

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$f_{xx} = 2y$, $f_{yy} = 0$, $f_{xy} = 2(x + 1)$. $D(-3, 0) = -16 < 0$,
$D(1, 0) = -16 < 0$. Both saddle points.

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Example: Let $g(x) = x^2$. Find the absolute max and min for g on $[-1, 3]$. $g'(x) = 2x = 0$ at $x = 0$. Check $g(0) = 0$, $g(-1) = 1$, and $g(3) = 9$, so absolute min is 0, absolute max is 9.

Absolute max/min in $\mathbb{R}^2$

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Example: Let $f(x, y) = x^2 + y^2$. Find the absolute max and min of f on the domain $R : \{x^2 + y^2 \leq 4\}$. $\nabla f = \langle 2x, 2y\rangle$, so $(0, 0)$ is the only critical points. It is in R, and $f(0, 0) = 0$. On the boundary of R, $x^2 + y^2 = 4$, so absolute min is 0 and absolute max is 4.

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Check points (including critical point):

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4) Evaluate the function at all of these points and take the max/min.

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On $x = 1$, parametrized curve is $x(t) = 1$, $y(t) = t$, so $g(t) = f(x(t), y(t)) = -2t^2 + t$ has critical point $1/4$, so $(1, 1/4)$ is in our list.

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List is:

$$\begin{aligned} f(1/2,1/4) &= -1/8 \\ f(1/2,0) &= -1/4 \\ f(1,1/4) &= 1/8 \\ f(0,0) &= 0 \\ f(1,0) &= 0 \\ f(1,1) &= -1 \end{aligned}$$

so absolute max is 1/8 and absolute min is -1.