Integration in Polar coordinates (16.3)

July 5, 2020

Big Picture for today

Big Picture: Polar coordinates work well in regions which are sectors.

Polar coordinates

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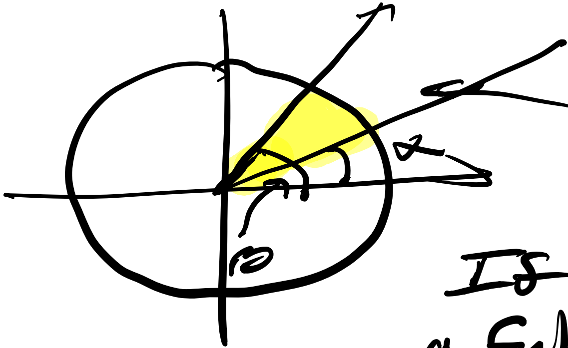
$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2 \rightsquigarrow r = \sqrt{x^2 + y^2},$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan(\theta),$$

so $\theta = \arctan(y/x)$.

Integration in polar coordinates

A *sector* of a disc is $0 \le r \le R$ and $\alpha \le \theta \le \beta$.



in radians

$$\text{Area} = \frac{(\beta - \alpha)}{2} r^2$$

If $\alpha = 0$, $\beta = 2\pi$ we have a full circle

$$\text{Area} = \frac{(2\pi - 0)}{2} r^2 = \pi r^2$$

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If $\Delta r_i = r_i - r_{i-1}$ and $\Delta\theta_j = \theta_j - \theta_{j-1}$, area of region $r_{i-1} \leq r_i$, $\theta_{j-1} \leq \theta \leq \theta_j$ is

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$\rightsquigarrow$

$$\iint_D f dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Circle example

Example: If D is the disc of radius R centered at $(0,0)$, then $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$.

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Example: If D is the disc of radius R centered at $(0, 0)$, then $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$. Area is

$$\iint_D 1\, dA = \int_0^{2\pi} \int_0^R (1)\, r\, dr\, d\theta = \pi R^2$$

Recall this was the last example from the 16.2 slides

Example

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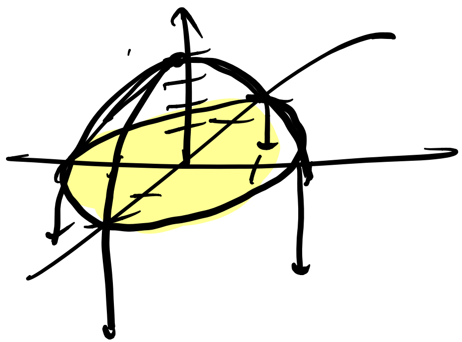
$$
\begin{aligned}
2\iint_D f dA &= 2\int_0^{2\pi}\int_0^R \sqrt{R^2 - r^2\cos^2\theta - r^2\sin^2\theta} d\theta \\
&= 2\int_0^{2\pi}\int_0^R \sqrt{R^2 - r^2}\, r\, dr\, d\theta \\
&= \frac{4}{3}\pi R^3
\end{aligned}
$$

Exercise: Find the volume under the paraboloid $f(x, y) = 4 - x^2 - y^2$ over the disc $x^2 + y^2 \leq 4$.



In polar coordinates

$$f(r,\theta) = f(r\cos\theta, r\sin\theta)$$
$$= 4 - r^2\cos^2\theta - r^2\sin^2\theta$$
$$= 4 - r^2$$

$$x^2+y^2 \leq 4 \Rightarrow 0 \leq r \leq 2$$

$$\iint_R f\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{1}{4} r^4 \right]_0^2 d\theta$$

$$= \int_0^{2\pi} [8 - 4] d\theta = 4\theta \Big|_0^{2\pi} = \boxed{8\pi}$$

Exercise: Find the volume under the paraboloid $f(x, y) = 4 - x^2 - y^2$ over the disc $x^2 + y^2 \leq 4$. Disc is $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$, so volume is

$$\int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta = 8\pi.$$

Exercise

Find the volume enclosed by $z = \frac{10}{1+x^2+y^2} - 2$ and the xy-plane.

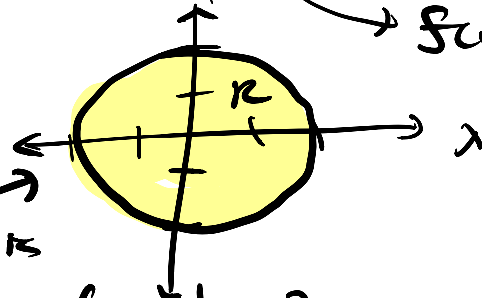
$$0 = \frac{10}{1+x^2+y^2} - 2 \Rightarrow$$

$$2 + 2(x^2+y^2) = 10$$

$$\Rightarrow 2(x^2+y^2) = 8$$

$$x^2+y^2 = 4$$

region is the circle of radius 2.



$$f(r,\theta) = \frac{10}{1+r^2} - 2$$

$$\text{Vol} = \int_0^{2\pi}\int_0^2 \left(\frac{10}{1+r^2} - 2\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} 5\ln(1+r^2) - r^2 \Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} 5\ln(5) - 2^2 \, d\theta = 2\pi(5\ln(5) - 4)$$

Exercise

Find the volume enclosed by $z = \frac{10}{1+x^2+y^2} - 2$ and the xy-plane. When $f = 0$, $1 + x^2 + y^2 = 5$, or $x^2 + y^2 = 4$. $\rightsquigarrow$ $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

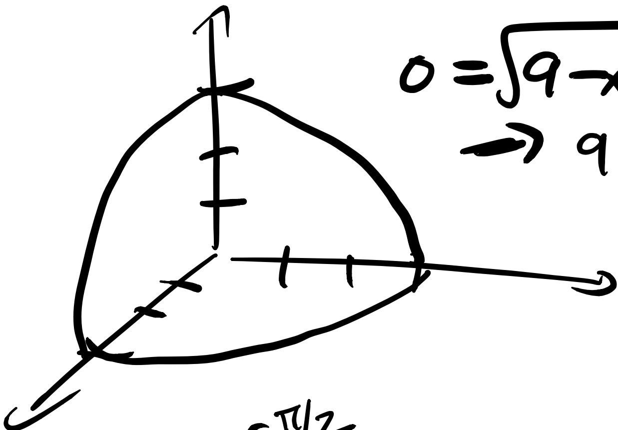
Exercise

Find the volume enclosed by $z = \frac{10}{1+x^2+y^2} - 2$ and the xy-plane. When $f = 0$, $1 + x^2 + y^2 = 5$, or $x^2 + y^2 = 4$. $\rightsquigarrow$ $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Volume is

$$\begin{aligned} &\int_0^{2\pi} \int_0^2 (\frac{10}{1+r^2} - 2) r \, dr \, d\theta \\ &= \int_0^{2\pi} (5 \ln(1 + r^2) - r^2)|_0^2 d\theta \\ &= \int_0^{2\pi} (5 \ln(5) - 4) d\theta \\ &= 2\pi(5 \ln(5) - 4). \end{aligned}$$

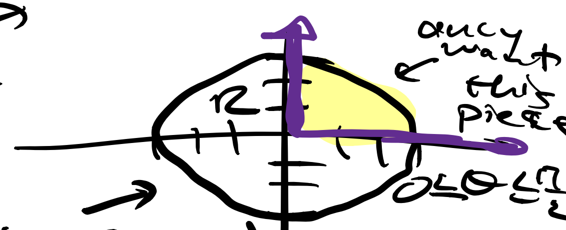
Example

Find the volume of the solid enclosed by $z = \sqrt{9 - x^2 - y^2}$ in the first octant.



$0 = \sqrt{9 - x^2 - y^2}$ sphere

$\rightarrow 9 = x^2 + y^2 \rightarrow$ circle of radius 3



$$\int_0^{\pi/2} \int_0^3 \sqrt{9 - r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{3}(9 - r^2)^{3/2} \Big|_0^3 \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3} 9^{3/2} \, d\theta = \frac{\frac{\pi}{2} \cdot 27}{3} = \boxed{9\pi/2}$$

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Volume is

$$\begin{aligned}
&\int_0^{\pi/2} \int_0^3 \sqrt{9 - r^2} r \, dr \, d\theta \\
&= \int_0^{\pi/2} (-(9 - r^2)^{3/2}/3)\Big|_0^3 d\theta \\
&= \int_0^{\pi/2} 9 d\theta \\
&= 9\pi/2.
\end{aligned}$$

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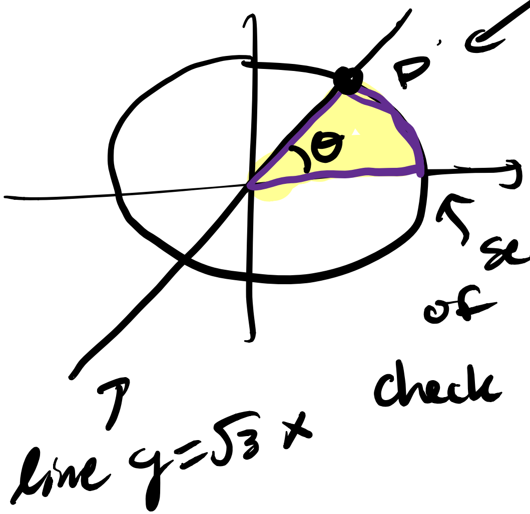
Should be 1/8th a sphere:

$$\frac{1}{8}(4\pi r^3/3) = 9\pi/2.$$

Exercise

Let R be the region $0 \leq x^2 + y^2 \leq 16$ and $0 \leq y \leq \sqrt{3}x$.
Evaluate the integral

$$\iint_R \frac{x + 10y}{1 + x^2 + y^2}\, dA.$$



sector of a circle of radius 4

check intersection point

using $y = \sqrt{3}x$

$x^2 + (\sqrt{3}x)^2 = 16$

$4x^2 = 16$

$x^2 = 4 \Rightarrow x = 2$

$\Rightarrow P: (2, 2\sqrt{3})$

$0 \leq \theta \leq \arctan\left(\frac{2\sqrt{3}}{2}\right)$

$\Rightarrow 0 \leq \theta \leq \pi/3$

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$$\iint_R \frac{x + 10y}{1 + x^2 + y^2} dA.$$

$0 \leq r \leq 4$ and $0 \leq \theta \leq \pi/3$. Integral is

$$\int_0^{\pi/3} \int_0^4 \frac{r\cos\theta + 10r\sin\theta}{1 + r^2} r\, dr\, d\theta.$$

$$\int_0^4 \int_0^{\pi/3} \frac{r^2}{1+r^2}(\cos\theta + 10\sin\theta)\, d\theta\, dr$$

$$= \int_0^4 \frac{r^2}{1+r^2}(\sin\theta - 10\cos\theta)\Big|_0^{\pi/3} dr$$

$$= \int_0^4 \frac{r^2}{1+r^2}\left(\left(\frac{\sqrt{3}}{2} - \frac{10}{2}\right) - (0 - 10)\right) dr$$

$$= \int_0^4 \left(\frac{\sqrt{3}+10}{2}\right) \frac{r^2}{1+r^2} dr$$

Hard to integrate

try $\frac{1+r^2-1}{1+r^2}$

$$= \left(\frac{\sqrt{3}+10}{2}\right) \int_0^4 \frac{r^2}{1+r^2} dr$$

$$= \frac{\sqrt{3}+10}{2} \int_0^4 \frac{1+r^2}{1+r^2} - \frac{1}{1+r^2} dr$$

$$= \frac{\sqrt{3}+10}{2} \Big[r - \arctan(r) \Big]_0^4$$

$$= \frac{\sqrt{3}+10}{2} \Big[4 - \arctan(4) \Big]$$

Exercise

Let R be the region $0 \le x^2 + y^2 \le 16$ and $0 \le y \le \sqrt{3}x$. Evaluate the integral

$$\iint_R \frac{x + 10y}{1 + x^2 + y^2} dA.$$

$0 \le r \le 4$ and $0 \le \theta \le \pi/3$. Integral is

$$\int_0^{\pi/3} \int_0^4 \frac{r\cos\theta + 10r\sin\theta}{1 + r^2} r \, dr \, d\theta.$$

$$\frac{r^2\cos\theta + 10r^2\sin\theta}{1 + r^2} = \frac{r^2}{1 + r^2}(\cos\theta + 10\sin\theta)$$

$$= \frac{1 + r^2 - 1}{1 + r^2}(\cos\theta + 10\sin\theta)$$

$$= (1 - \frac{1}{1 + r^2})(\cos\theta + 10\sin\theta).$$

Continued

$$\int_0^{\pi/3}\int_0^4 \left((1-\frac{1}{1+r^2})(\cos\theta+10\sin\theta)\right) drd\theta$$

$$= \int_0^{\pi/3} (r-\arctan(r))(\cos\theta+10\sin\theta)|_0^4 d\theta$$

$$= \int_0^{\pi/3} (4-\arctan(4))(\cos\theta+10\sin\theta) d\theta$$

$$= (4-\arctan(4))(\sin\theta-10\cos\theta)|_0^{\pi/3}$$

$$= (4-\arctan(4))((\sqrt{3}/2-5)-(-10)).$$

Example

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$$\iint_R (1)\, dA$$

using polar coordinates.

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or

$$r^2 - 2r\sin\theta = 0.$$

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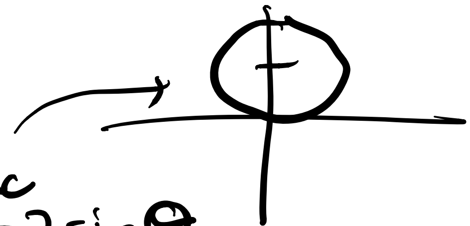

using polar coordinates. From precalc $r = 2\sin\theta$

$$\begin{aligned} 1 = x^2 + (y-1)^2 &= r^2\cos^2\theta + (r\sin\theta - 1)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1, \end{aligned}$$

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Rearrange: $r^2 = 2r\sin\theta$, which has two solutions: $r = 0$ and $r = 2\sin\theta$.

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Rearrange: $r^2 = 2r\sin\theta$, which has two solutions: $r = 0$ and $r = 2\sin\theta$. $0 \le \theta \le \pi$.

Continued

$$\iint_R dA = \int_0^{\pi}\int_0^{2\sin\theta} r dr d\theta$$
$$= \int_0^{\pi} \frac{1}{2} r^2 |_0^{2\sin\theta} d\theta$$
$$= \int_0^{\pi} 2\sin^2\theta d\theta$$
$$= \int_0^{\pi} 2(\frac{1}{2} - \frac{1}{2}\cos(2\theta)) d\theta$$
$$= \pi$$

Exercise

Let R be the part of the disc of radius 2 centered at the point $(0, -2)$ in the fourth quadrant ($x \geq 0,\ y \leq 0$). Compute the integral

$$\iint_R x dA.$$

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$$\iint_R x\,dA. = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{-4\sin\theta} r\cos\theta \, r\,dr\,d\theta$$

$$4 = x^2 + (y+2)^2 = r^2 + 4r\sin\theta + 4,$$

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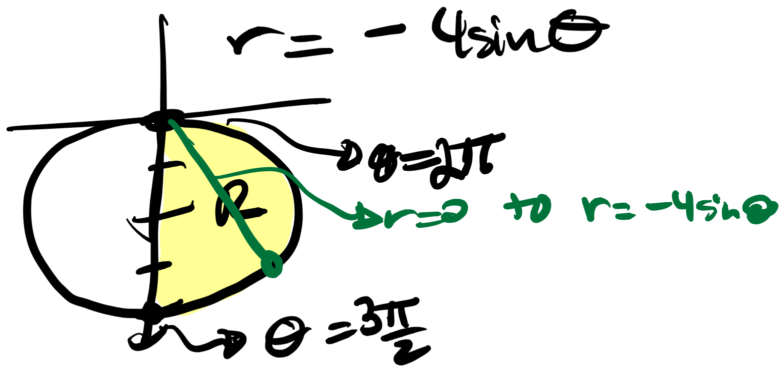

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Fourth quadrant $\rightsquigarrow 3\pi/2 \leq \theta \leq 2\pi$. $\rightsquigarrow \sin\theta \leq 0$, so domain is $0 \leq r \leq -4\sin\theta$, $3\pi/2 \leq \theta \leq 2\pi$.

$$\iint_R x dA = \int_{3\pi/2}^{2\pi} \int_0^{-4\sin\theta} r\cos\theta \, (r \, dr d\theta)$$

$$= \int_{3\pi/2}^{2\pi} \left[\frac{1}{3} r^3 \cos\theta \right]_0^{-4\sin\theta} d\theta$$

$$= \int_{3\pi/2}^{2\pi} -\frac{64}{3} \sin^3\theta \cos\theta \, d\theta$$

$$= \left[-\frac{16}{3} \sin^4\theta \, d\theta \right]_{3\pi/2}^{2\pi}$$

$$= 0 - \left(-\frac{16}{3} (-1)^4 \right) = \boxed{\frac{16}{3}}$$

Continued

$$\iint_R x dA = \int_{3\pi/2}^{2\pi} \int_0^{-4\sin\theta} r\cos(\theta) r dr d\theta$$
$$= \int_{3\pi/2}^{2\pi} (1/3) r^3 |_0^{-4\sin\theta} \cos\theta d\theta$$
$$= \int_{3\pi/2}^{2\pi} (1/3)(-64)\sin^3\theta\cos\theta d\theta$$
$$= (-16/3)\sin^4\theta |_{3\pi/2}^{2\pi}$$
$$= 16/3.$$