Triple Integrals (16.4)

July 11, 2020

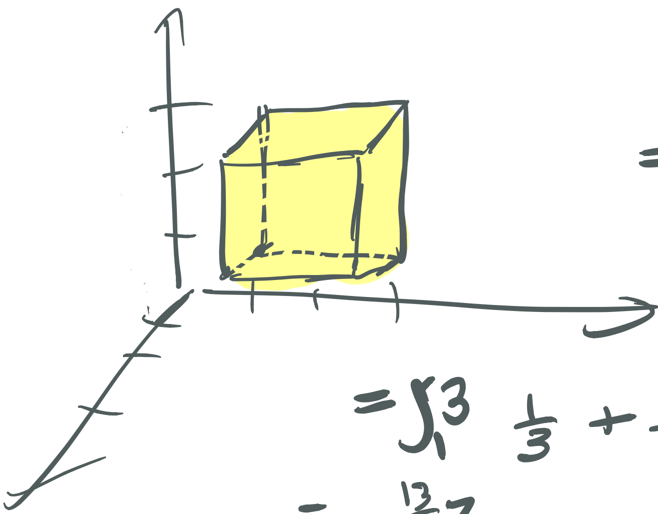
Big Picture for today

Big Picture: Triple integrals work just like double integrals.

Example

Example: Let $D = [0, 1] \times [1, 2] \times [1, 3]$ be a rectangular parallelepiped (a box) and $f(x, y, z) = x^2 + xy - z$. Compute

$$\iiint_D f dV.$$



$$= \int_1^3 \int_1^2 \int_0^1 x^2 + xy - z \, dx\, dy\, dz$$

$$= \int_1^3 \int_1^2 \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 y - zx \Big|_0^1 dy\, dz$$

$$= \int_1^3 \int_1^2 \tfrac{1}{3} + \tfrac{1}{2}y - z \, dy\, dz$$

$$= \int_1^3 \tfrac{1}{3}y + \tfrac{1}{4}y^2 - zy \Big|_1^2 dy\, dz$$

$$= \int_1^3 \tfrac{1}{3} + \tfrac{3}{4} - z \, dz$$

$$= \tfrac{13}{12}z - \tfrac{1}{2}z^2 \Big|_1^3 = \left[\tfrac{13 \cdot 3}{12} - \tfrac{9}{2}\right] - \left[\tfrac{13}{12} + \tfrac{1}{2}\right]$$

Example

$= \frac{2 \cdot 13}{12} - 4 = \frac{13}{6} - 4$

$= \frac{13}{6} - \frac{24}{6}$

$= -\frac{11}{6}$

Example: Let $D = [0, 1] \times [1, 2] \times [1, 3]$ be a rectangular parallelepiped (a box) and $f(x, y, z) = x^2 + xy - z$. Compute

$$\iiint_D f dV.$$

$$\iiint_D f dV = \int_0^1 \int_1^2 \int_1^3 (x^2 + xy - z) dz\, dy\, dx = 2/3 + 3/2 - 4.$$

Example

Example: Let $D = [0,1] \times [1,2] \times [1,3]$ be a rectangular parallelepiped (a box) and $f(x,y,z) = x^2 + xy - z$. Compute

$$\iiint_D f dV.$$

$$\iiint_D f dV = \int_0^1 \int_1^2 \int_1^3 (x^2 + xy - z) dz\, dy\, dx = 2/3 + 3/2 - 4.$$

Note: Fubini still works and you can change the order of integration if it helps.

Exercise

Exercise: Let $D = [1,2] \times [0,2] \times [1,2]$ and $f(x,y,z) = 2xz + 3z^2y - 6y$. Compute $\iiint_D f dV$.

Exercise: Let $D = [1,2] \times [0,2] \times [1,2]$ and $f(x,y,z) = 2xz + 3z^2y - 6y$. Compute $\iiint_D f dV$.

$$\begin{aligned}
\iiint_D f dV &= \int_1^2 \int_0^2 \int_1^2 (2xz + 3z^2y - 6y) dz\, dy\, dx \\
&= \int_1^2 \int_0^2 (xz^2 + z^3y - 6yz)|_1^2 dy\, dx \\
&= \int_1^2 \int_0^2 (3x + y) dy\, dx \\
&= \int_1^2 (3xy + y^2/2)|_0^2 dx \\
&= \int_1^2 (6x + 2) dx \\
&= 11
\end{aligned}$$

Exercise

Let $D = [1,2] \times [1,3] \times [0,2]$ and $f(x,y,z) = x + y + 2xz$.
Compute $\iiint_D f dV$.

Exercise

Let $D = [1,2] \times [1,3] \times [0,2]$ and $f(x,y,z) = x + y + 2xz$. Compute $\iiint_D f dV$.

$$\begin{aligned}
&\int_1^2 \int_1^3 \int_0^2 (x + y + 2xz) dz\, dy\, dx \\
&= \int_1^2 \int_1^3 (xz + yz + xz^2)|_0^2 dy\, dx \\
&= \int_1^2 \int_1^3 (2x + 2y + 4x) dy\, dx \\
&= \int_1^2 (6xy + y^2)|_1^3 dx \\
&= \int_1^2 ((18x + 9) - (6x + 1)) dx \\
&= (6x^2 + 8x)|_1^2 = 26.
\end{aligned}$$

In 1-d, $\int_a^b 1\,dx = (b-a)$ is the *length* of the interval $[a, b]$.

Volume vs integral

In 1-d, $\int_a^b 1\,dx = (b - a)$ is the *length* of the interval $[a, b]$.
$\int_a^b f\,dx$ is the *area* of the region under f.

Volume vs integral

In 1-d, $\int_a^b 1\,dx = (b - a)$ is the *length* of the interval $[a, b]$.
$\int_a^b f\,dx$ is the *area* of the region under f.
In 2-d, $\iint_R 1\,dA$ is the *area* of the region R.

Volume vs integral

In 1-d, $\int_a^b 1\,dx = (b - a)$ is the *length* of the interval $[a, b]$. $\int_a^b f\,dx$ is the *area* of the region under f.
In 2-d, $\iint_R 1\,dA$ is the *area* of the region R. $\iint_R f\,dA$ is the *volume* of the region over R and under f.

Volume vs integral

In 1-d, $\int_a^b 1 dx = (b - a)$ is the *length* of the interval $[a, b]$.
$\int_a^b f\, dx$ is the *area* of the region under f.
In 2-d, $\iint_R 1 dA$ is the *area* of the region R. $\iint_R f\, dA$ is the *volume* of the region over R and under f.
In 3-d, $\iiint_D 1 dV$ is the *volume* of the region D.

Volume vs integral

In 1-d, $\int_a^b 1\,dx = (b-a)$ is the *length* of the interval $[a, b]$.
$\int_a^b f\,dx$ is the *area* of the region under f.
In 2-d, $\iint_R 1\,dA$ is the *area* of the region R. $\iint_R f\,dA$ is the *volume* of the region over R and under f.
In 3-d, $\iiint_D 1\,dV$ is the *volume* of the region D. $\iiint_D f\,dV$ is the 4-dimensional volume of the region "over D" and "under f".

Volume vs integral

In 1-d, $\int_a^b 1 dx = (b - a)$ is the *length* of the interval $[a, b]$.
$\int_a^b f\, dx$ is the *area* of the region under f.
In 2-d, $\iint_R 1 dA$ is the *area* of the region R. $\iint_R f\, dA$ is the *volume* of the region over R and under f.
In 3-d, $\iiint_D 1 dV$ is the *volume* of the region D. $\iiint_D f\, dV$ is the 4-dimensional volume of the region "over D" and "under f".
Interpretation: If f describes the *density* at each point of a solid, then $\iiint_D f\, dV$ is the mass.

Volume vs integral

In 1-d, $\int_a^b 1\,dx = (b - a)$ is the *length* of the interval $[a, b]$.
$\int_a^b f\,dx$ is the *area* of the region under f.
In 2-d, $\iint_R 1\,dA$ is the *area* of the region R. $\iint_R f\,dA$ is the *volume* of the region over R and under f.
In 3-d, $\iiint_D 1\,dV$ is the *volume* of the region D. $\iiint_D f\,dV$ is the 4-dimensional volume of the region "over D" and "under f".
Interpretation: If f describes the *density* at each point of a solid, then $\iiint_D f\,dV$ is the mass.
Another interpretation: If f gives the temperature at point (x, y, z), then $\iiint_D f\,dV$ measures total heat content.

Volume vs integral

In 1-d, $\int_a^b 1\,dx = (b-a)$ is the *length* of the interval $[a,b]$.
$\int_a^b f\,dx$ is the *area* of the region under f.
In 2-d, $\iint_R 1\,dA$ is the *area* of the region R. $\iint_R f\,dA$ is the *volume* of the region over R and under f.
In 3-d, $\iiint_D 1\,dV$ is the *volume* of the region D. $\iiint_D f\,dV$ is the 4-dimensional volume of the region "over D" and "under f".
Interpretation: If f describes the *density* at each point of a solid, then $\iiint_D f\,dV$ is the mass.
Another interpretation: If f gives the temperature at point (x,y,z), then $\iiint_D f\,dV$ measures total heat content.
Another interpretation: 4-dimensional "volume" could be volume of space-time.

More general regions

Recall: In 2d, we integrated $a \leq x \leq b$, $g(x) \leq y \leq h(x)$. Integrate in y first gives a function of x, then integrate in x.

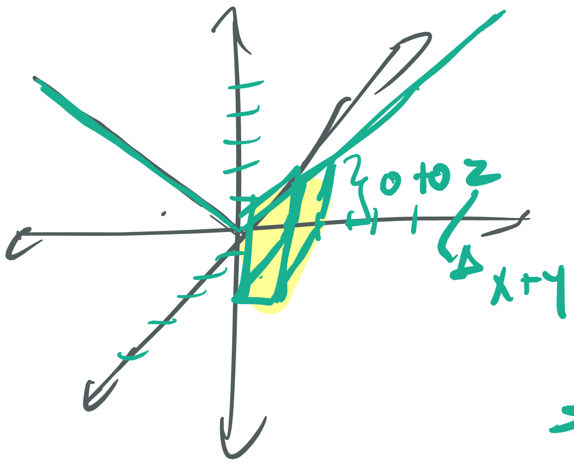
More general regions

Recall: In 2d, we integrated $a \leq x \leq b$, $g(x) \leq y \leq h(x)$. Integrate in y first gives a function of x, then integrate in x. In 3d, can integrate $a \leq x \leq b$, $g(x) \leq y \leq h(x)$, *and* $G(x, y) \leq z \leq H(x, y)$. Integrate in z first gives a function of x and y. Then integrate in y, giving a function of x, then integrate in x.

More general regions

Recall: In 2d, we integrated $a \leq x \leq b$, $g(x) \leq y \leq h(x)$. Integrate in y first gives a function of x, then integrate in x. In 3d, can integrate $a \leq x \leq b$, $g(x) \leq y \leq h(x)$, *and* $G(x, y) \leq z \leq H(x, y)$. Integrate in z first gives a function of x and y. Then integrate in y, giving a function of x, then integrate in x. With each integration there should be one fewer variables.

Example: Find the volume of the region D under the plane $z = x + y$ and over the rectangle $[0,3] \times [1,2]$ in the xy-plane.



$$\iiint_R 1\,dV$$

$$\int_0^3 \int_1^2 \int_0^{x+y} 1\,dz\,dy\,dx$$

$$= \int_0^3 \int_1^2 x+y\,dy\,dx$$

$$= \int_0^3 xy + \tfrac{1}{2}y^2 \Big|_1^2 dy = \int_0^3 x + \tfrac{3}{2}\,dx$$

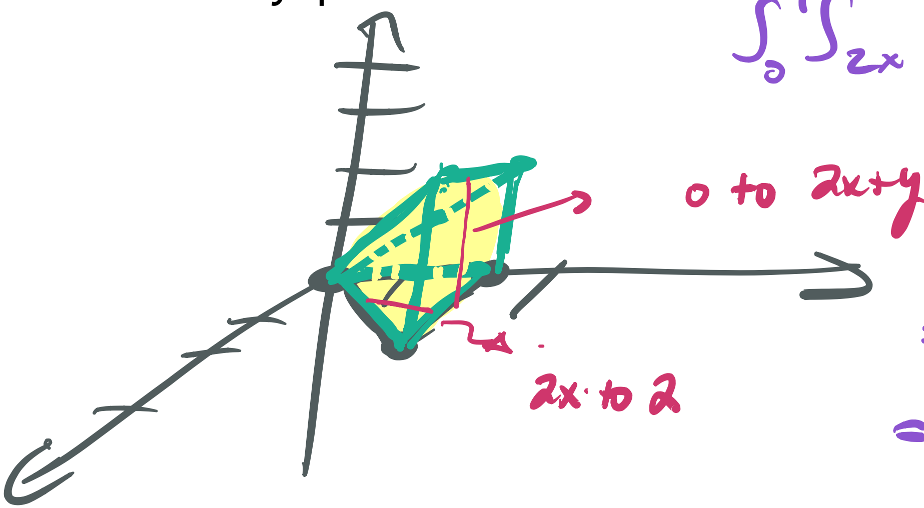
$$= \tfrac{1}{2}x^2 + \tfrac{3}{2}x \Big|_0^3 = \tfrac{9}{2} + \tfrac{9}{2} - 0 = 18/2 = 9$$

Example

Example: Find the volume of the region D under the plane $z = x + y$ and over the rectangle $[0,3] \times [1,2]$ in the xy-plane. Volume is by integrating the function $f(x,y,z) = 1$:

$$\iiint_D 1\,dV = \int_0^3 \int_1^2 \int_0^{x+y} 1\,dz\,dy\,dx = 9$$

Exercise: Find the volume of the region under the plane $z = 2x + y$ and over the triangle with vertices $(0,0), (0,2), (1,2)$ in the xy-plane.



$$\int_0^1\int_{2x}^{2}\int_0^{2x+y} 1\,dz\,dy\,dx$$

$$= \int_0^1\int_{2x}^{2} 2x+y\,dy\,dx$$

$$= \int_0^1 2xy+\tfrac{1}{2}y^2\Big|_{2x}^{2}\,dx$$

$$= \int_0^1 (4x+2)-(4x^2+2x^2)\,dx$$

$$= \int_0^1 -6x^2+4x+2\,dx$$

$$= -\tfrac{6}{3}x^3+\tfrac{4}{2}x^2+2x\Big|_0^1$$

$$= -2+2+2=2$$

Exercise

Exercise: Find the volume of the region under the plane $z = 2x + y$ and over the triangle with vertices $(0,0), (0,2), (1,2)$ in the xy-plane.

Triangle is $0 \le x \le 1$ and $2x \le y \le 2$, so Volume is

$$\int_0^1 \int_{2x}^2 \int_0^{2x+y} 1 \, dz \, dy \, dx$$
$$= \int_0^1 \int_{2x}^2 (2x + y) dy \, dx$$
$$= \int_0^1 (2xy + y^2/2)\big|_{2x}^2 dx$$
$$= \int_0^1 (-6x^2 + 4x + 2) dx$$
$$= (-2x^3 + 2x^2 + 2x)\big|_0^1$$
$$= 2.$$

Exercise

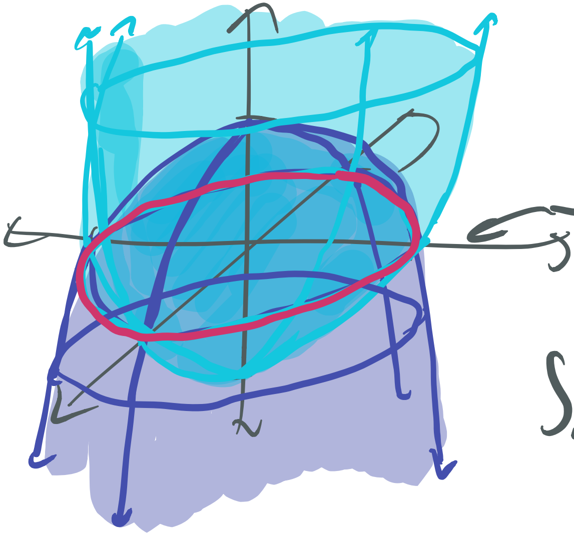
Compute $\int_1^2 \int_x^{3x} \int_0^{3+x-y} 2xy \, dz \, dy \, dx$.

Exercise

Compute $\int_1^2 \int_x^{3x} \int_0^{3+x-y} 2xy\, dz\, dy\, dx$.

$$\begin{aligned}
&\int_1^2 \int_x^{3x} \int_0^{3+x-y} 2xy\, dz\, dy\, dx \\
&= \int_1^2 \int_x^{3x} (2xyz)|_0^{3+x-y} dy\, dx \\
&= \int_1^2 \int_x^{3x} (6xy + 2x^2y - 2xy^2) dy\, dx \\
&= \int_1^2 (3xy^2 + x^2y^2 - (2/3)xy^3)|_x^{3x} dx \\
&= \int_1^2 (24x^3 - (28/3)x^4) dx \\
&= (6x^4 - (28/15)x^5)|_1^2 \\
&= (6(2^4) - (28/15)(2^5)) - (6 - (28/15))
\end{aligned}$$

Exercise: Let D be the region enclosed by the paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2 - 1$ and $f(x, y, z) = xy - 3z$. Compute $\iiint_D f dV$.



$1 - x^2 - y^2 = x^2 + y^2 - 1$

$2 = (x^2 + y^2)2$

$1 = x^2 + y^2$

intersect at $x^2 + y^2 = 1$ and $z = 0$.

$$\iiint_D f\,dV = \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2-1}^{x^2+y^2+1} xy - 3z \, dz\,dy\,dx$$

Exercise

Exercise: Let D be the region enclosed by the paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2 - 1$ and $f(x, y, z) = xy - 3z$. Compute $\iiint_D f dV$.

$$-1+x^2+y^2 \leq z \leq 1-x^2-y^2, \; -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \; -1 \leq x \leq 1$$

Exercise

Exercise: Let D be the region enclosed by the paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2 - 1$ and $f(x, y, z) = xy - 3z$. Compute $\iiint_D f dV$.

$$-1+x^2+y^2 \le z \le 1-x^2-y^2, \ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \ -1 \le x \le 1$$

$$\begin{aligned}\iiint_D f dV &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-1+x^2+y^2}^{1-x^2-y^2}(xy-3z)dz\,dy\,dx \\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(xyz-3z^2/2)\Big|_{-1+x^2+y^2}^{1-x^2-y^2}dy\,dx \\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}2xy(1-x^2-y^2)dy\,dx.\end{aligned}$$

Continued

Now switch to polar: $x = r\cos\theta$, $y = r\sin\theta$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$

Continued

Now switch to polar: $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xy(1-x^2-y^2)dy\,dx$$
$$= \int_0^{2\pi}\int_0^1 2r^2\cos(\theta)\sin(\theta)(1-r^2)r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 2\cos(\theta)\sin(\theta)(r^3-r^5)\,dr\,d\theta$$
$$= \int_0^{2\pi} 2\cos(\theta)\sin(\theta)(r^4/4-r^6/6)|_0^1 d\theta$$
$$= (1/4-1/6)\int_0^{2\pi} 2\cos(\theta)\sin(\theta)d\theta$$
$$= (1/4-1/6)\sin^2(\theta)|_0^{2\pi} = 0.$$