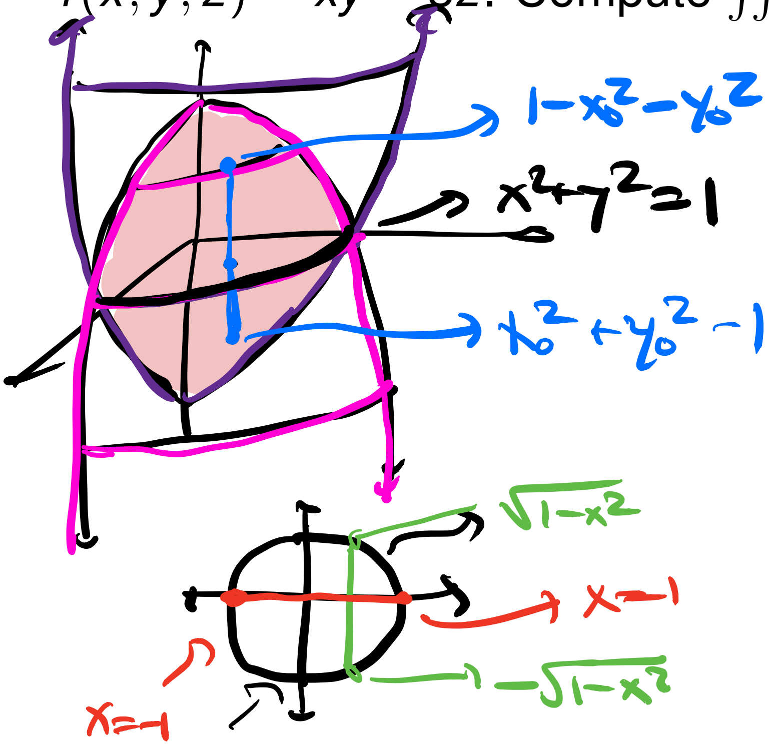
Triple Integrals in Cylindrical and Spherical Coordinates (16.5)

July 11, 2020

Big Picture for today

Big Picture: Cylindrical coordinates are polar in (x, y) and z. Spherical coordinates are more complicated.

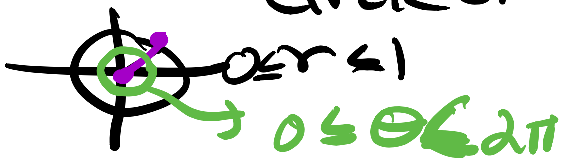
Example from last time: Let D be the region enclosed by the paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2 - 1$ and $f(x, y, z) = xy - 3z$. Compute $\iiint_D f dV$.



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2-1}^{1-x^2-y^2} xy-3z\,dz\,dy\,dx$$

switch to polar.

circle of radius



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$$-1+x^2+y^2 \leq z \leq 1-x^2-y^2, \; -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \; -1 \leq x \leq 1$$

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$$-1+x^2+y^2 \le z \le 1-x^2-y^2, \; -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \; -1 \le x \le 1$$

$$\begin{aligned}
\iiint_D f dV &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-1+x^2+y^2}^{1-x^2-y^2} (xy - 3z) dz\, dy\, dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (xyz - 3z^2/2)\Big|_{-1+x^2+y^2}^{1-x^2-y^2} dy\, dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xy(1 - x^2 - y^2) dy\, dx.
\end{aligned}$$

just a double integral so switch to polar

circle of radius 1

$\to r^2$

$$= \int_0^{2\pi} \int_0^1 2(r\cos\theta)(r\sin\theta)(1-r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 2r^3 \cos\theta\sin\theta - 2r^5 \cos\theta\sin\theta\, dr\,d\theta$$

$$\tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

$$= \int_0^{2\pi} \left[\frac{2}{4}r^4 - \frac{2}{6}r^6\right](\cos\theta\sin\theta)\Big|_0^1 \,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}\cos\theta\sin\theta\, d\theta \qquad u = \sin\theta, \quad du = \cos\theta$$

$$= \frac{1}{6}\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{2\pi} = \frac{1}{12}\left[\underbrace{\sin^2(2\pi)}_{0} - \underbrace{\sin^2(0)}_{0}\right]$$

$$= \boxed{0}.$$

Continued

Now switch to polar: $x = r\cos\theta$, $y = r\sin\theta$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$

Continued

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$$\begin{aligned}
&\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xy(1-x^2-y^2)dy\,dx \\
&= \int_0^{2\pi}\int_0^1 2r^2\cos(\theta)\sin(\theta)(1-r^2)r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 2\cos(\theta)\sin(\theta)(r^3-r^5)r\,dr\,d\theta \\
&= \int_0^{2\pi} 2\cos(\theta)\sin(\theta)(r^4/4-r^6/6)|_0^1 d\theta \\
&= (1/4-1/6)\int_0^{2\pi}\cos(\theta)\sin(\theta)d\theta \\
&= (1/4-1/6)\sin^2(\theta)|_0^{2\pi} = 0.
\end{aligned}$$

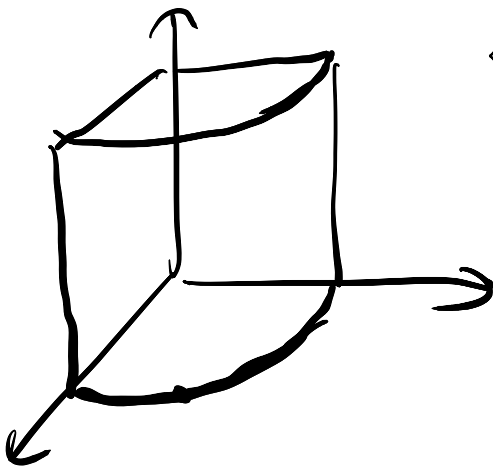
Cylindrical coordinates

Definition: Cylindrical coordinates are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

polar coordinate

$dV = r\,dr\,d\theta\,dz$



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That is, polar coordinates in (x, y) and z stays the same. z stays the same, so

$$dV = dx\,dy\,dz = rdr\,d\theta\,dz.$$

Back to example

$$f(x,y,z) = xy - 3z$$

$$f(r,\theta,z) = r\cos\theta\, r\sin\theta - 3z$$

Let D be the region enclosed by the paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2 - 1$. Write in cylindrical coordinates.

$$r^2 = x^2 + y^2$$

$z = 1 - r^2$ top z-value

$z = r^2 - 1$ bottom z value

$$\int_0^{2\pi}\int_0^1\int_{r^2-1}^{1-r^2} (r^2\cos\theta\sin\theta - 3z)\, r \, dz\, dr\, d\theta$$

Back to example

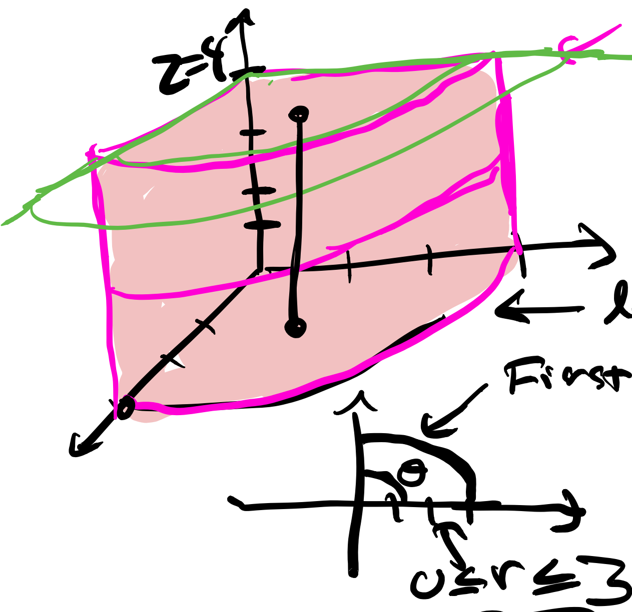
Let D be the region enclosed by the paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2 - 1$. Write in cylindrical coordinates.

$$0 \leq \theta \leq 2\pi, \; 0 \leq r \leq 1, \; -1 + r^2 \leq z \leq 1 - r^2.$$

Use cylindrical coordinates to describe the part of the cylinder of radius 3 in the first octant and bounded by the plane $z = 4$.

cylinder of radius 3

$0 \le \theta \le \pi/2, 0 \le r \le 3, 0 \le z \le 4$



lower bound is the xy-plane $z=0$

First quadrant piece of a circle of radius 3

$0 \le \theta \le \pi/2$

$0 \le z \le 4$

Example

Use cylindrical coordinates to describe the part of the cylinder of radius 3 in the first octant and bounded by the plane $z = 4$. First octant means $x \geq 0$, $y \geq 0$, $z \geq 0$.

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Use cylindrical coordinates to describe the part of the cylinder of radius 3 in the first octant and bounded by the plane $z = 4$. First octant means $x \geq 0$, $y \geq 0$, $z \geq 0$. Radius 3 means that $0 \leq r \leq 3$ (in the (x, y) plane). Angle $0 \leq \theta \leq \pi/2$. Height $z = 4$, so

$$0 \leq r \leq 3, \; 0 \leq \theta \leq \pi/2, \; 0 \leq z \leq 4.$$

General regions

If $f(x, y, z)$ is a nice function and D is a region in $\mathbb{R}^3$, have (x, y) in polar, and $G(x, y) \leq z \leq H(x, y)$.

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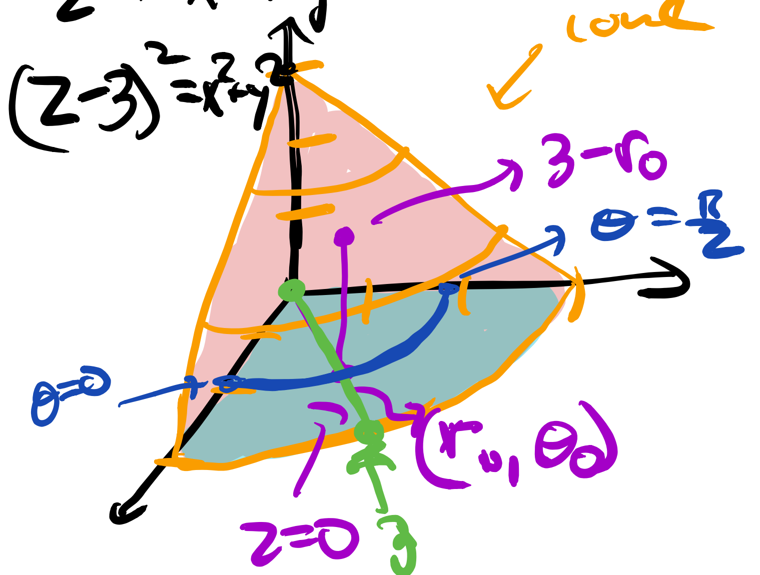
$$\rightsquigarrow \iiint_D f dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) dz\, r dr\, d\theta$$

Exercise

→ cone is like $z=y$, $z=x$ rotated

Let D be the quarter cone in the first octant of radius 3 and height 3, opening downward. Let $f = 2xz$. Convert to cylindrical coordinates and compute $\iiint_D f dV$. $2r\cos\theta z$



$(z-3)^2 = x^2+y^2$

$z=3$ $x,y=0$

$z=0$ $x^2+y^2=3$

$z = 3 \pm \sqrt{x^2+y^2}$

$z = 3-\sqrt{x^2+y^2}$

$z=3-r$

$\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{3-r} 2r\cos\theta z \, dz \, r \, dr \, d\theta$

$$\int_0^{\pi/2}\int_0^3 r^2\cos\theta\, z^2\Big|_0^{3-r}\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^3 r^2\cos\theta\,(3-r)^2\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^3 (9r^2 - 6r^3 + r^4)\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[3r^3 - \frac{6}{4}r^4 + \frac{1}{5}r^5\right]\cos\theta\Big|_0^3\,d\theta$$

$$= \int_0^{\pi/2}\left[3^4 - \frac{6}{4}\cdot 3^4 + \frac{1}{5}3^5\right]\cos\theta\,d\theta$$

$$= \left[3^4 - \frac{6}{4}3^4 + \frac{3^5}{5}\right]\sin\theta\Big|_0^{\pi/2}$$

$$= \boxed{3^4 - \frac{6}{4}3^4 + \frac{3^5}{5}} = 81 + \frac{3^5}{5} - \frac{3^5}{2}$$

Exercise

Let D be the quarter cone in the first octant of radius 3 and height 3, opening downward. Let $f = 2xz$. Convert to cylindrical coordinates and compute $\iiint_D f dV$.
$0 \leq r \leq 3$, $0 \leq \theta \leq \pi/2$. $0 \leq z \leq 3 - r$.

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Let D be the quarter cone in the first octant of radius 3 and height 3, opening downward. Let $f = 2xz$. Convert to cylindrical coordinates and compute $\iiint_D f dV$.
$0 \le r \le 3$, $0 \le \theta \le \pi/2$. $0 \le z \le 3 - r$.

$$\iiint_D f dV = \int_0^{\pi/2} \int_0^3 \int_0^{3-r} 2zr\cos\theta \, dz\, rdr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 z^2 r\cos\theta \Big|_0^{3-r} \, rdr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 (3-r)^2 r\cos\theta, \, rdr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 (9r^2 + r^4 - 6r^3)\cos\theta \, dr\, d\theta$$

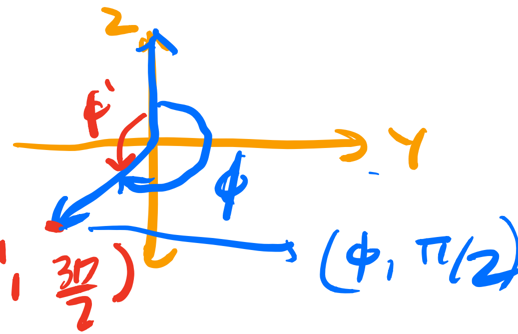
Continued

$$= \int_0^{\pi/2} (3r^3 + r^5/5 - 3r^4/2)|_0^3 \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (81 + (3^5)/5 - (3^5)/2) \cos\theta \, d\theta$$

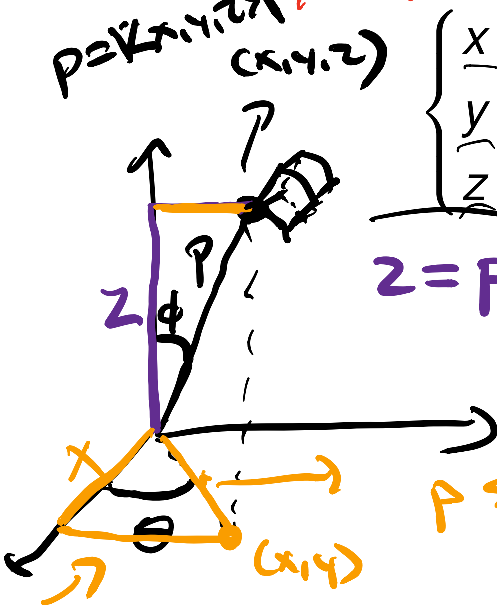
$$= (81 + (3^5)/5 - (3^5)/2).$$

Spherical coordinates



$$\rho^2 = x^2 + y^2 + z^2$$

Definition:



$$\begin{cases} x = \rho \sin\phi \cos\theta, \\ y = \rho \sin\phi \sin\theta, \\ z = \rho \cos\phi \end{cases}$$

$0 \le \rho$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$z = \rho \cos\phi$

$\rho \sin\phi$

Volume is a small part of a sphere

$x = \cos\theta(\rho\sin\phi)$ $\quad$ $y = \sin\theta(\rho\sin\phi)$

Definition: In spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example: Let D be the unit ball. Convert to spherical and compute

$$\iiint_D (x^2 + y^2 + z^2)^{5/2}\, dV.$$

$\to \rho^2$

$\int (x^2+y^2+z^2)^{5/2}\, dz$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^{5/2}\, dz\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho^2)^{5/2}\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^7 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{1}{8}\rho^8 \sin\phi \Big|_0^1 \, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{1}{8} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{8} \cos\phi \Big|_0^{\pi} d\theta$$

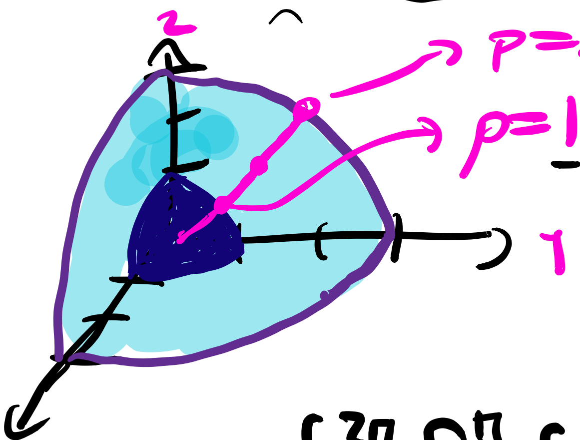
$$= \int_0^{2\pi} \frac{1}{8} - \left(-\frac{1}{8}\right) d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \pi/2$$

Example: Let D be the unit ball. Convert to spherical and compute

$$\iiint_D (x^2+y^2+z^2)^{5/2}\,dV.$$

$$\begin{aligned}
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^5\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} (1/8)\sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi} (1/8)(1-(-1))d\theta \\
&= \pi/2.
\end{aligned}$$

Exercise: Let D be the region between the spheres centered at $(0,0,0)$ of radius 1 and radius 3. Use spherical coordinates to compute $\iiint_D (x^2+y^2+z^2)^2 dV$.



$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$\int_1^3 = 9-1=8$

$4-0=4$

$$\int_0^{2\pi}\int_0^{\pi}\int_1^3 (\rho^2)^2 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^3 \rho^6 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^7}{7} \sin\phi \Big|_1^3 \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{3^7}{7} - \frac{1}{7}\right) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{3^7}{7} - \frac{1}{7}\right)\left(\underbrace{-\cos\phi}_{2}\right)\Big|_0^{\pi} d\theta$$

$$= \int_0^{2\pi} 2\left(\frac{3^7}{7} - \frac{1}{7}\right) d\theta$$

$$= \boxed{4\pi\left(\frac{3^7}{7} - \frac{1}{7}\right)} = \frac{8744\pi}{7}$$

Exercise

Exercise: Let D be the region between the spheres centered at $(0,0,0)$ of radius 1 and radius 3. Use spherical coordinates to compute $\iiint_D (x^2+y^2+z^2)^2 dV$.

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^3 \rho^4\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (\rho^7/7)|_1^3 \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (3^7/7 - 1/7) \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi} (3^7/7 - 1/7)(-\cos\phi)|_0^{\pi} d\theta$$

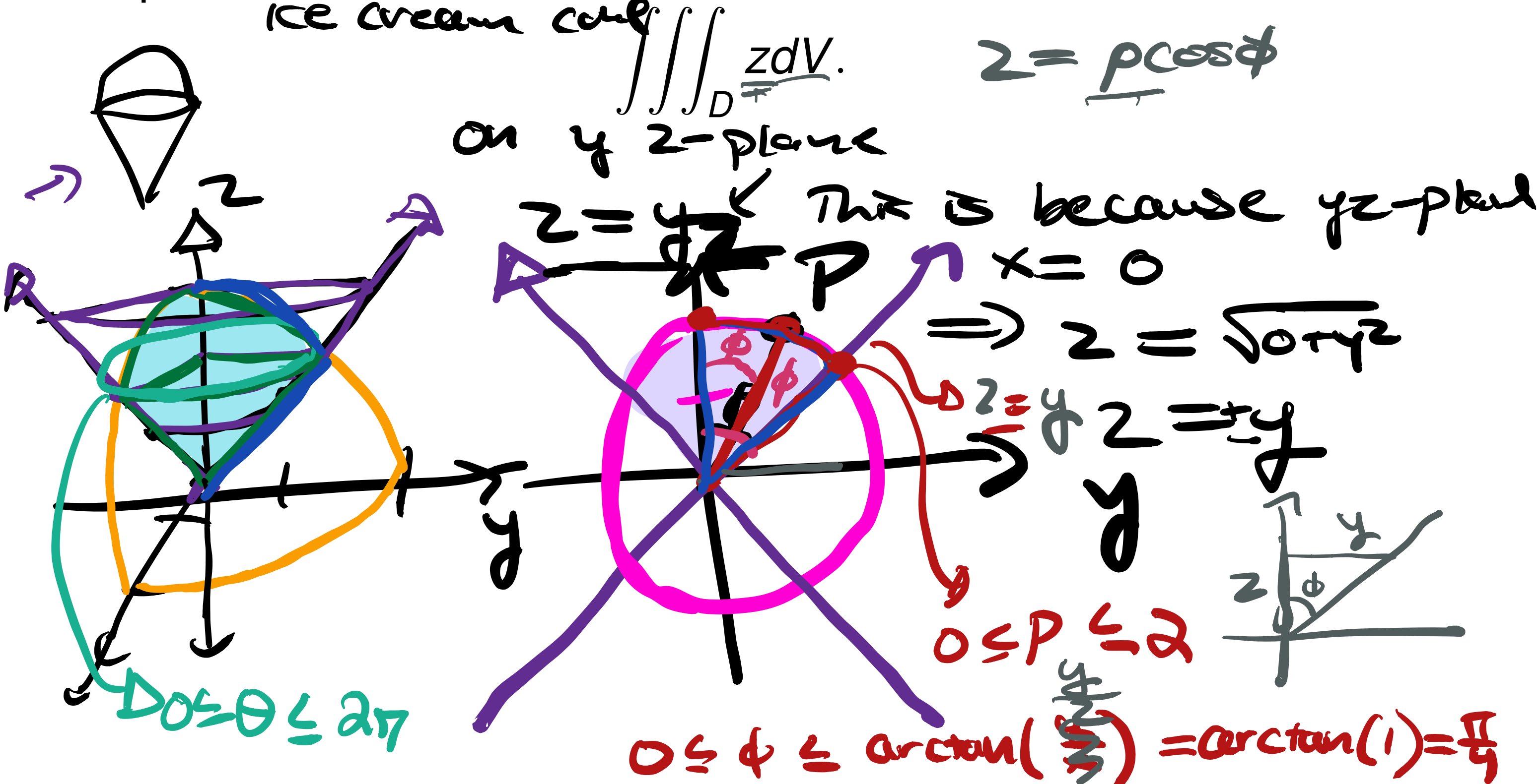
$$= \int_0^{2\pi} 2(3^7/7 - 1/7) d\theta$$

$$= 4\pi(3^7/7 - 1/7).$$

Example

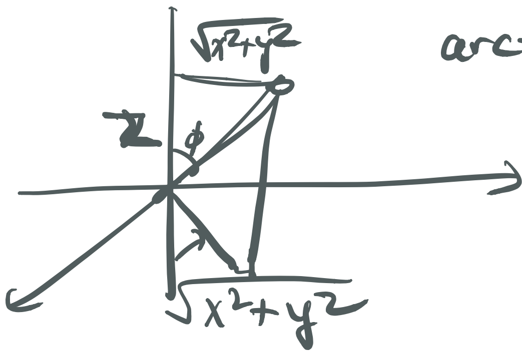
Example: Let D be the region bounded by the sphere of radius 2 and above the cone $z = \sqrt{x^2 + y^2}$. Convert to spherical and compute

$$\iiint_D z\,dV.$$



$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho\cos\phi\,[\rho^2\sin\phi\, d\rho\, d\phi\, d\theta]$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^3 \cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$



$\arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \phi$

$$\int_0^{2\pi}\int_0^{\pi/4} \frac{1}{4}\rho^4\cos\phi\sin\phi \Big|_0^2 d\phi\, d\theta$$

$u = \cos\phi$

$u = \sin\phi$

$$= \int_0^{2\pi}\int_0^{\pi/4} 4\cos\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} 2\sin^2\phi \Big|_0^{\pi/4} d\theta$$

$\sin \pi/4 = \sqrt{2}/2$

$2/4 = \frac{1}{2}$

$$= \int_0^{2\pi} 1\, d\theta$$

$$= 2\pi$$

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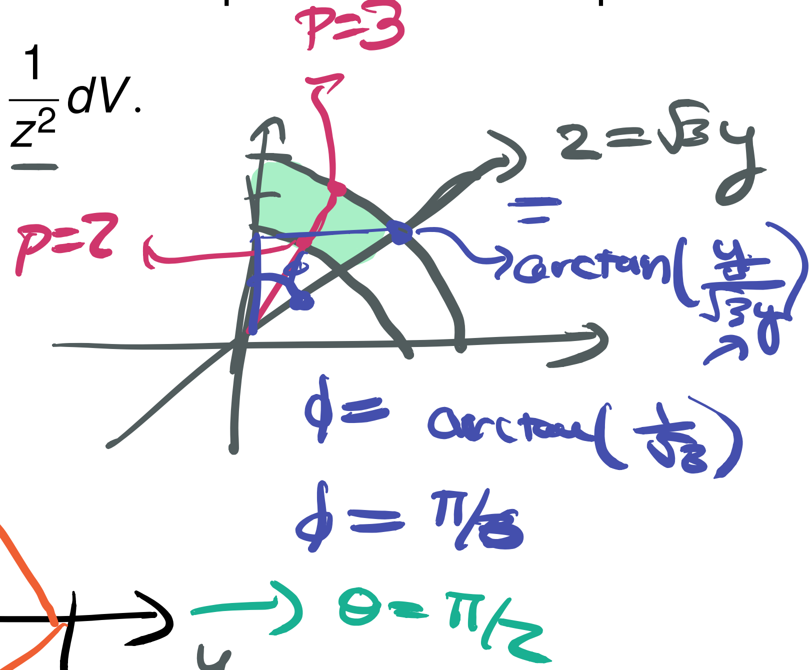
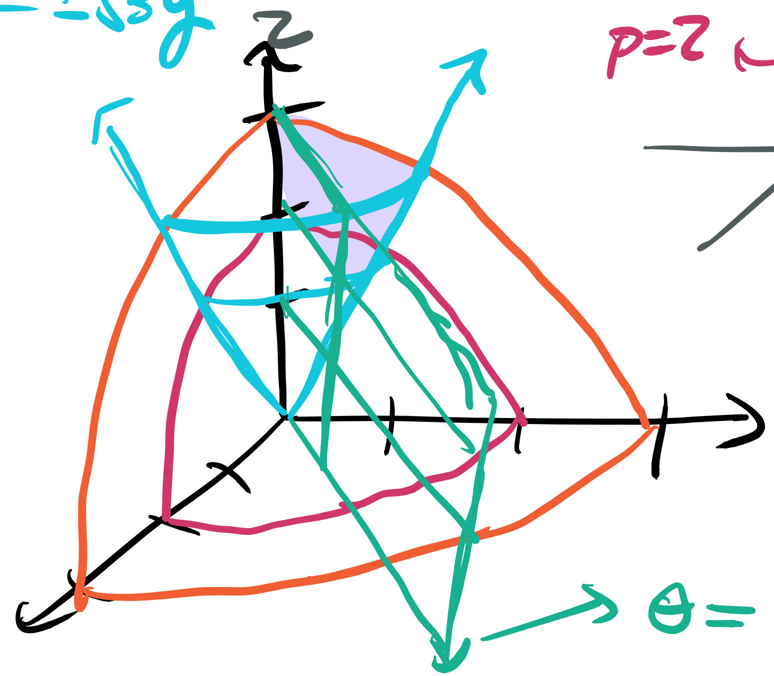
$$\iiint_D z dV.$$

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho \cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} (\rho^4/4)|_0^2 \sin\phi \cos\phi \, d\phi \, d\theta \\
&= 4 \int_0^{2\pi} (\frac{1}{2} \sin^2\phi)|_0^{\pi/4} d\theta \\
&= 2\pi.
\end{aligned}$$

Exercise: Let D be the region in the in the first octant between the spheres of radius 2 and 3, above the cone $z = \sqrt{3x^2 + 3y^2}$, and $y \geq x$. Convert to spherical and compute

$$\iiint_D \frac{1}{z^2}\, dV.$$





$$\int_{\pi/4}^{\pi/2}\int_{0}^{\pi/8}\int_{2}^{3} \frac{1}{\cancel{\rho^2}\cos^2\phi}\,\cancel{\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$z = \rho\cos\phi$$

$$z^2 = \rho^2\cos^2\phi$$

Exercise

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$$\iiint_D \frac{1}{z^2} dV.$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{\pi/6} \int_2^3 \frac{1}{\rho^2 \cos^2 \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{\pi/6} \rho|_2^3 \frac{\sin \phi}{\cos^2 \phi} d\phi \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{\cos \phi}|_0^{\pi/6} d\theta$$

$$= \int_{\pi/4}^{\pi/2} (1/(\sqrt{3}/2) - 1) d\theta = (\pi/4)(2/\sqrt{3} - 1).$$

Example

Example:
Convert to polar and evaluate:

$$\int_0^2 \int_0^{(4-x^2)^{1/2}} \int_{(x^2+y^2)^{1/2}}^{(8-x^2-y^2)^{1/2}} (x^2+y^2+z^2)^2 dz\, dy\, dx.$$

Example:
Convert to polar and evaluate:

$$\int_0^2 \int_0^{(4-x^2)^{1/2}} \int_{(x^2+y^2)^{1/2}}^{(8-x^2-y^2)^{1/2}} (x^2+y^2+z^2)^2 dz\, dy\, dx.$$

$$= \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{2\sqrt{2}} \rho^6 \sin\phi d\rho\, d\phi\, d\theta$$

Continued

$$= \int_0^{\pi/2} \int_0^{\pi/4} (\rho^7/7)|0^{2\sqrt{2}} \sin\phi d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/4} ((2\sqrt{2})^7/7) \sin\phi d\phi\, d\theta$$

$$= \int_0^{\pi/2} ((2\sqrt{2})^7/7)(-\cos\phi)_0^{\pi/4} d\theta$$

$$= \int_0^{\pi/2} ((2\sqrt{2})^7/7)(1 - 1/\sqrt{2}) d\theta$$

$$= 2\pi((2\sqrt{2})^7/7)(1 - 1/\sqrt{2}).$$