Line integrals (17.2)

July 15, 2020

Big Picture for today

Big Picture: A line integral generalizes a 1-d integral to curves.

Question: Let $\mathbf{r}(t)$ parametrize a curve in $\mathbb{R}^2$, and let $f(x, y)$ be a continuous function. How does one find the area under f *over* $\mathbf{r}(t)$?

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$$\int_C f ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt.$$

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Steps:

1) Find a parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$ of C.

2) Compute $|\mathbf{r}'(t)|$.

3) $\int_C f ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|dt$.

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$$\begin{aligned}
\int_C f ds &= \int_0^1 f(2+t, 1-2t)\sqrt{5}dt \\
&= \int_0^1 (2+t+2(1-2t))\sqrt{5}dt \\
&= \int_0^1 (4-3t)\sqrt{5}dt \\
&= (5/2)\sqrt{5}.
\end{aligned}$$

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$$\begin{aligned}\int_C fds &= \int_0^{2\pi} (3(3\cos t) + (3\sin t)^2)3dt \\ &= \int_0^{2\pi} (9\cos t + 9(\frac{1}{2} - \frac{1}{2}\cos 2t)3dt \\ &= (9\sin t + (9/2)t - (9/4)\sin(2t))3|_0^{2\pi} \\ &= 27\pi.\end{aligned}$$

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Note: Could also use $\mathbf{r}(t) = \left\langle 3\cos(t^2), 3\sin(t^2) \right\rangle$, $0 \le t \le \sqrt{2\pi}$, so $|\mathbf{r}'(t)| = 3t$. (Exercise)

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$$\begin{aligned}\int_C f ds &= \int_0^2 2t(1 + 4t^2)^{\frac{1}{2}} dt \\ &= (1 + 4t^2)^{3/2}/6|_0^2 \\ &= ((17)^{3/2} - 1)/6.\end{aligned}$$

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Line integrals in $\mathbb{R}^3$

Recall: In $\mathbb{R}^2$, ds was computed using Pythagorean theorem on $\mathbf{r}'$ to get $ds = |\mathbf{r}'(t)|$. $\rightsquigarrow$ Pythagorean theorem in $\mathbb{R}^3$, so $ds = |\mathbf{r}'|$. **Theorem:** Let C be a curve in $\mathbb{R}^3$ parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$, and let $f(x, y, z)$ a continuous function which is continuous near C. Then

$$\int_C f ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt.$$

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$$\int_C f ds = \int_0^1 (1+2t)(1)\sqrt{5} dt$$
$$= \sqrt{5}(t+t^2)|_0^1 = 2\sqrt{5}.$$

Exercise

Exercise: Let C be the helix parametrized by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$, $0 \leq t \leq 2\pi$. Let $f(x, y, z) = y - z$. Compute $\int_C f ds$.

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$$\begin{aligned}\int_C f ds &= \int_0^{2\pi} (3\sin t - 4t)5dt \\ &= (-3\cos t - 2t^2)(5)\Big|_0^{2\pi} \\ &= -10(2\pi)^2.\end{aligned}$$

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$$\begin{aligned} \int_C x ds &= \int_0^1 t^3 (9t^4 + 16)^{1/2} dt \\ &= (9t^4 + 16)^{3/2} (2/3)(1/36)|_0^1 \\ &= ((5)^3 - 4^3)(2/3)(1/36) \end{aligned}$$

Vector fields and "work"

If $\mathbf{F}$ is a vector field (on $\mathbb{R}^2$ or $\mathbb{R}^3$) and C is a curve parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$, then the component of $\mathbf{F}$ in the direction of $\mathbf{r}(t)$ is

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where $\mathbf{T} = (\mathbf{r}')/|\mathbf{r}'|$ is the unit tangent vector.

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For a curve $\mathbf{r}(t)$, $a \leq t \leq b$, the work moving from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_0 + \Delta t)$ is (Work)$\simeq (\mathbf{F} \cdot \mathbf{T})|\mathbf{r}'(t_0)|\Delta t$.

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Summing all of these up over the curve $\mathbf{r}$ and taking $\Delta t \to 0$, we get

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F} \cdot (\mathbf{r}'(t)/|\mathbf{r}'(t)|)|\mathbf{r}'(t)|dt = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)dt.$$

Example

Let $\mathbf{F} = \langle x, y \rangle$, and let $\mathbf{r}(t) = \left\langle 4t, t^2 \right\rangle$, $0 \leq t \leq 2$. Compute $\int_C \mathbf{F} \cdot \mathbf{T} ds$.

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$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^2 \left\langle 4t, t^2 \right\rangle \cdot \langle 4, 2t \rangle \\
&= \int_0^2 (16t + 2t^3) dt \\
&= (8t^2 + (1/2)t^4)|_0^2 \\
&= (32 + (1/2)(16))
\end{aligned}$$

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$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^1 \left\langle t^4, t \right\rangle \cdot \langle 1, 2t \rangle \, dt \\
&= \int_0^1 (t^4 + 2t^2) dt \\
&= (t^5/5 + (2/3)t^3)|_0^1 \\
&= (1/5 + 2/3).
\end{aligned}$$

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$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^{2\pi} \langle -2\sin t, 2\cos t, t/2\pi \rangle \cdot \langle -2\sin t, 2\cos t, 1/2\pi \rangle \, dt \\
&= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t + t/(2\pi)^2) dt \\
&= (4t + t^2/(8\pi^2))|_0^{2\pi} \\
&= (8\pi + 1/2).
\end{aligned}$$

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\int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^1 \left\langle 2t^2, 4t^2 + 1 \right\rangle \cdot \langle 1, 1 \rangle \, dt \\
&= \int_0^1 (2t^2 + 4t^2 + 1) dt \\
&= (2t^3 + t)\Big|_0^1 \\
&= 3.
\end{aligned}$$

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Let $\mathbf{F} = \langle 2y^2, 4xy + 1 \rangle$, and let C be the part of the parabola $y = x^2$ between $(0, 0)$ and $(1, 1)$. Compute $\int_C \mathbf{F} \cdot \mathbf{T} ds$.

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&= \int_0^1 (2t^4 + 8t^4 + 2t) dt \\
&= (2t^5 + t^2)|_0^1 \\
&= 3.
\end{aligned}$$