Conservative vector fields (17.3)

July 15, 2020

Big Picture for today

Big Picture: Conservative vector fields lead to a fundamental theorem of calculus for line integrals.

Line integrals

Recall: If $\mathbf{F}$ is a vector field, and $\mathbf{r}(t)$, $a \leq t \leq b$ parametrizes a curve C, then the line integral

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

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$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^1 \left\langle t^4, t \right\rangle \cdot \langle 1, 2t \rangle \, dt \\ &= \int_0^1 (t^4 + 2t^2) dt \\ &= (t^5/5 + (2/3)t^3)|_0^1 \\ &= (1/5 + 2/3). \end{aligned}$$

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$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^1 \left\langle t^2, t \right\rangle \cdot \langle 1, 1 \rangle \, dt \\ &= \int_0^1 (t^2 + t) dt \\ &= 1/3 + 1/2 \\ &\neq 1/5 + 2/3\end{aligned}$$

from last example.

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$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^1 \left\langle 2t^2, 4t^2 + 1 \right\rangle \cdot \langle 1, 1 \rangle \, dt \\
&= \int_0^1 (2t^2 + 4t^2 + 1) dt \\
&= (2t^3 + t)|_0^1 \\
&= 3.
\end{aligned}$$

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$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T} ds &= \int_0^1 \left\langle 2t^4, 4t^3 + 1 \right\rangle \cdot \langle 1, 2t \rangle \, dt \\ &= \int_0^1 (2t^4 + 8t^4 + 2t) dt \\ &= (2t^5 + t^2)|_0^1 \\ &= 3.\end{aligned}$$

It would be nice to know when a line integral is independent of path.

Conservative vector fields

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Definition: Let R be a region in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Let **F** be a vector field on R. If there exists a function $\phi(x, y)$ (or $\phi(x, y, z)$) on R such that $\mathbf{F} = \nabla\phi$, then **F** is *conservative*.

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How do you determine if it is conservative? If $\phi(x, y)$ is a scalar function with 2 continuous derivatives, then $\phi_{xy} = \phi_{yx}$. So if $\mathbf{F} = \langle f, g\rangle$ satisfies $\mathbf{F} = \nabla\phi = \langle \phi_x, \phi_y\rangle$, then $f_y = \phi_{xy}$ and $g_x = \phi_{yx}$.

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In $\mathbb{R}^3$, there are 3 mixed partials, xy, xz, yz, so if $\mathbf{F} = \langle f, g, h\rangle$ is $\nabla\phi$, then $f_y = g_x$, $f_z = h_x$, and $g_z = h_y$.

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Theorem: Let R be a simply connected domain in $\mathbb{R}^2$ and $\mathbf{F} = \langle f, g \rangle$ a vector field on R, with f and g having continuous first partial derivatives. Then $\mathbf{F}$ is conservative if and only if $f_y = g_x$.

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Theorem: Let R be a simply connected domain in $\mathbb{R}^2$ and $\mathbf{F} = \langle f, g \rangle$ a vector field on R, with f and g having continuous first partial derivatives. Then $\mathbf{F}$ is conservative if and only if $f_y = g_x$. Let R be a simply connected domain in $\mathbb{R}^3$ and $\mathbf{F} = \langle f, g, h \rangle$ be a vector field on R with f, g, h having continuous first partials. Then $\mathbf{F}$ is conservative if and only if $f_y = g_x$, $f_z = h_x$, and $g_z = h_y$.

Example

Let $\mathbf{F} = \langle z + 4xy, 2x^2, x + 3 \rangle$. Determine if **F** is conservative.

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$$f_y = 4x, \;\; g_x = 4x,$$

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$$f_y = 2x, \; g_x = 0,$$

so not conservative.

Finding potentials

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$$\phi = 2y^2 x + c(y).$$

$$\phi_y = 4xy + c'(y) = 4xy + 1 \implies c'(y) = 1.$$

So $c(y) = y$, and $\phi = 2y^2 x + y$.

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$$\phi_x = z + 4xy \implies \phi = xz + 2x^2 y + c(y, z),$$

$$\phi_y = 2x^2 + c_y = 2x^2 \implies c_y = 0$$

or $c(y, z) = d(z)$, so $\phi = xz + 2x^2 y + d(z)$.

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or $c(y, z) = d(z)$, so $\phi = xz + 2x^2y + d(z)$.

$$\phi_z = x + d' = x + 3 \implies d' = 3,$$

or $d = 3z$, and $\phi = xz + 2x^2y + 3z$.

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$$\phi(x, y) = xy^2 + 2x^2 - 4y.$$

Curl

Recall: If $\mathbf{F} = \langle f, g, h \rangle$ is defined on a simply connected set, then $\mathbf{F}$ is conservative if and only if

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Another way to remember is *curl*:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \\ &= (h_y - g_z)\mathbf{i} - (h_x - f_z)\mathbf{j} + (g_x - f_y)\mathbf{k}. \end{aligned}$$

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$$= (h_y - g_z)\mathbf{i} - (h_x - f_z)\mathbf{j} + (g_x - f_y)\mathbf{k}.$$

$\mathbf{F}$ conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$

Warmup

Exercise: Determine if the following vector fields are conservative or not. If conservative, find a potential function.

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$$\int_a^b u(x)dx = U(b) - U(a) = \phi(b, 0) - \phi(a, 0).$$

On the other hand:

$$\begin{aligned}\int \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F} \cdot \mathbf{r}'(t)dt \\ &= \int_0^1 \langle u(\langle a, 0 \rangle + t \langle b - a, 0 \rangle), 0 \rangle \cdot \langle b - a, 0 \rangle \, dt \\ &= \int_0^1 u(a + t(b - a))(b - a)dt = \int_a^b u(x)dx.\end{aligned}$$

FTC for line integrals

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“Proof”:

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"Proof": Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Chain rule:

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"Proof": Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Chain rule:

$$\begin{aligned} \frac{d}{dt}\phi(\mathbf{r}(t)) &= \frac{d}{dt}\phi(x(t), y(t)) \\ &= \phi_x \frac{dx}{dt} + \phi_y \frac{dy}{dt} \\ &= \langle \phi_x, \phi_y \rangle \cdot \langle dx/dt, dy/dt \rangle \\ &= \nabla\phi \cdot \mathbf{r}'. \end{aligned}$$

Independence of path

Corollary: Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let $\phi : R \to \mathbb{R}$ be a differentiable function. Let $\mathbf{r}_1 = \langle x_1(t), y_1(t) \rangle$ and $\mathbf{r}_2 \langle x_2(t), y_2(t) \rangle$, $a \leq t \leq b$ be two paths with $\mathbf{r}_1(b) = \mathbf{r}_2(b)$ and $\mathbf{r}_1(a) = \mathbf{r}_2(a)$, then

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$$\int \nabla\phi \cdot d\mathbf{r}_1 = \int \nabla\phi \cdot d\mathbf{r}_2.$$

$\rightsquigarrow$ Can choose a path which is convenient.

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$$\int \mathbf{F} \cdot d\mathbf{r}_1 = \int_{-1}^{1} \langle 4t, -4 \rangle \cdot \langle 1, 0 \rangle \, dt = \int_{-1}^{1} 4t dt.$$

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$$\int \mathbf{F} \cdot d\mathbf{r}_2 = \int_{-1}^{1} \left\langle (1 - t^2)^2 + 4t, 2t(1 - t^2) - 4 \right\rangle \cdot \langle 1, -2t \rangle \, dt$$
$$= \int_{-1}^{1} ((1 - t^2)^2 + 4t - 4t^2(1 - t^2) + 8t) dt$$

Exercise

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Note: To compute directly is

$$\int_0^\pi \left\langle \sin t - 3t^2, \cos t, -6t \cos t + 4 \right\rangle \cdot \langle -\sin t, \cos t, 1 \rangle \, dt.$$