Dot and Cross product

May 29, 2020

Big Picture for today

Big Picture: 1) The dot product eats two vectors and spits out a scalar. This number says how much of one vector lives in the direction of the other.
2) The cross product eats two vectors and spits out another vector. This vector is perpendicular to the other two.

Angle between unit vectors

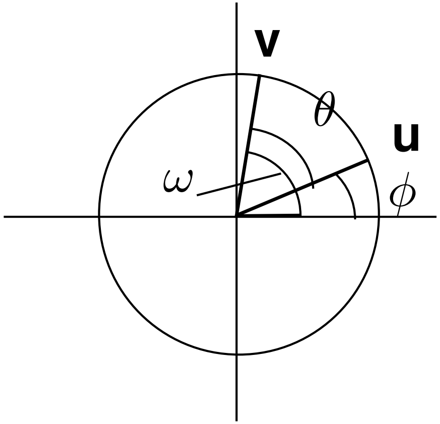

Figure: Unit vectors and their angles.

u has angle ϕ, **v** has angle ω so angle between is $\theta = \omega - \phi$.

Angle between unit vectors

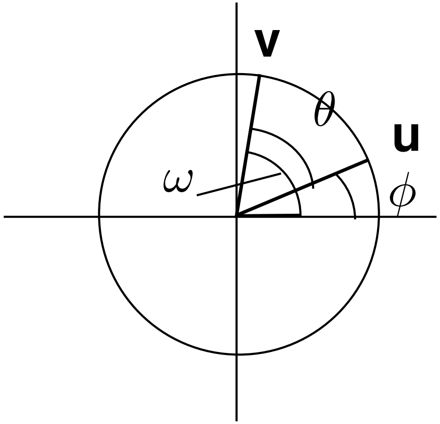

Figure: Unit vectors and their angles.

u has angle ϕ, **v** has angle ω so angle between is $\theta = \omega - \phi$.

$$\mathbf{u} = \langle \cos\phi, \sin\phi \rangle, \ \mathbf{v} = \langle \cos\omega, \sin\omega \rangle .$$

Angle between cont'd

Note:

$$\cos(\theta) = \cos(\omega - \phi) = \cos(\omega)\cos(-\phi) - \sin(\omega)\sin(-\phi)$$

Angle between cont’d

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$$\cos(\theta) = \cos(\omega - \phi) = \cos(\omega)\cos(-\phi) - \sin(\omega)\sin(-\phi)$$

$\cos$ even, $\sin$ odd

$$\rightsquigarrow \cos(\theta) = \cos(\omega)\cos(\phi) + \sin(\omega)\sin(\phi).$$

Angle between cont'd

Note:

$$\cos(\theta) = \cos(\omega - \phi) = \cos(\omega)\cos(-\phi) - \sin(\omega)\sin(-\phi)$$

cos even, sin odd

$$\rightsquigarrow \cos(\theta) = \underset{x_2}{\cos(\omega)}\,\underset{x_1}{\cos(\phi)} + \underset{y_2}{\sin(\omega)}\,\underset{y_1}{\sin(\phi)}.$$

So if $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$ are unit vectors,

$$\cos(\theta) = x_1 x_2 + y_1 y_2. = u \cdot v$$

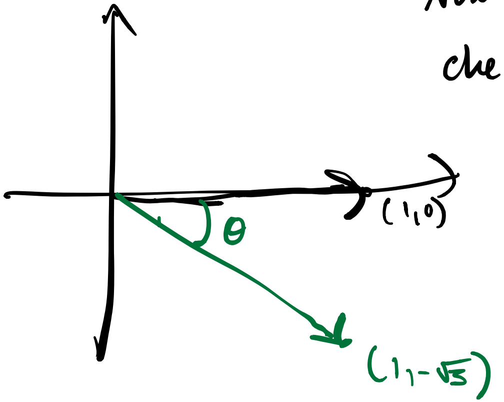
Note that if u, v are not unit vectors then

$$\cos(\theta) = \frac{x_1}{|\vec{u}|}, \quad \sin(\theta) = \frac{y_1}{|\vec{u}|}, \quad \cos(\omega) = \frac{x_2}{|\vec{v}|} \quad \sin(\omega) = \frac{y_2}{|\vec{v}|}$$

So, $\cos\theta = \frac{x_1}{|\vec{u}|}\frac{x_2}{|\vec{v}|} + \frac{y_1}{|\vec{u}|}\frac{y_2}{|\vec{v}|} \Rightarrow |\vec{u}|\,|\vec{v}|\cos(\theta) = x_1x_2 + y_1y_2$

Examples

Example: Let **u** $= \langle 1, 0 \rangle$ and **v** $= \left\langle 1, -\sqrt{3} \right\rangle$. Find angle between.



Now we can use the above but let's check it use special angles first

Remember from pre-calc $\cos(-60^\circ) = 1/2$

$\sin(-60^\circ) = -\sqrt{3}/2$

So the angle from $\langle 1, -\sqrt{3} \rangle$ to the positive x-axis is 60° and $\langle 1, 0 \rangle$ is along the x-axis so $\theta = 60^\circ$

Using the above formula, $|u| = 1$ $|v| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

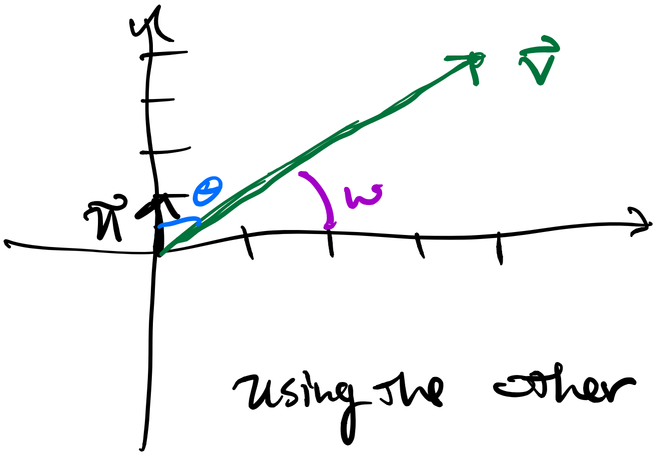
$$\cos(\theta) = \frac{1}{1} \cdot \frac{1}{2} + \frac{0}{1} \cdot \frac{-\sqrt{3}}{2} = \frac{1}{2}$$

$$\cos(\theta) = 1/2 \rightarrow \theta = \cos^{-1}(1/2) \Rightarrow \theta = 60^\circ$$

Examples

Example: Let $\mathbf{u} = \langle 1, 0\rangle$ and $\mathbf{v} = \left\langle 1, -\sqrt{3}\right\rangle$. Find angle between.

Exercise: Let $\mathbf{u} = \langle 0, 1\rangle$, $\mathbf{v} = \langle 4, 4\rangle$. Find angle between.



Remember from precalc

$\sin(45^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}$

so $w = 45^\circ$.

since $\vec{u}$ is along the positive y-axis $\theta = 90^\circ - w = \boxed{45^\circ}$

Using the other method $|u| = 1$ $|v| = 4\sqrt{2}$

$$\cos(\theta) = \frac{0}{1}\,\frac{4}{4\sqrt{2}} + \frac{1}{1}\,\frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\sqrt{2}/2\right) = \boxed{45^\circ}$$

$\mathbb{R}^3$ and a convenient formula

Harder to visualize “angle between” in $\mathbb{R}^3$. But in general, if **u** and **v** are two position vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$),

Definition of dot product:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$$

where θ is angle between.

We will show a nice formula for $\vec{u} \cdot \vec{v}$ later.

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where θ is angle between.

Question: What if $\mathbf{v} = \mathbf{0}$? $|\vec{0}| = 0$ so $\vec{u} \cdot \vec{0} = |\vec{u}| \cdot 0 \cos(\theta)$

$= 0$

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where θ is angle between.

Question: What if $\mathbf{v} = \mathbf{0}$?

Question: What if $\mathbf{u} \perp \mathbf{v}$? $\longrightarrow$ If $\vec{u} \perp \vec{v}$ then $\theta = 90^\circ$ so

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(90^\circ) = |\vec{u}||\vec{v}| \cdot 0 = 0.$$

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Definition of dot product:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$$

so $\vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2 + z_1z_2$ even if it's rotated.

where θ is angle between.

Question: What if $\mathbf{v} = \mathbf{0}$?

Question: What if $\mathbf{u} \perp \mathbf{v}$?

Theorem: Let $\mathbf{u} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2, z_2 \rangle$ be two vectors. Then

$$\mathbf{u} \cdot \mathbf{v} = x_1x_2 + y_1y_2 + z_1z_2.$$

Rotated in $y,z \to$ plane

$$\vec{u} = \langle x_1, \underline{y_1\cos\theta - z_1\sin\theta}, \underline{y_1\sin\theta + z_1\cos\theta} \rangle$$

$$\to \vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2\cos^2\theta - y_1z_2\sin\theta\cos\theta - z_1y_2\sin\theta\cos\theta + z_1z_2\sin^2\theta$$
$$+ y_1y_2\sin^2\theta + y_1z_2\sin\theta\cos\theta + y_2z_1\sin\theta\cos\theta + z_1z_2\cos^2\theta$$

$= y_1y_2$

$= 0$

$= 0$

z_1z_2 //

Proof idea

Book uses law of cosines. Here is another idea.

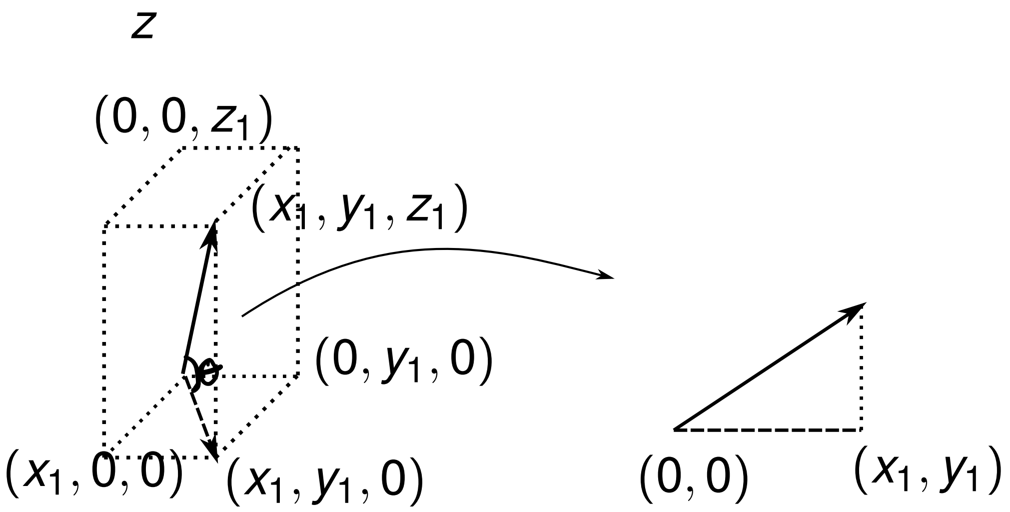

Figure: The triangle with bottom edge in *xy* plane.

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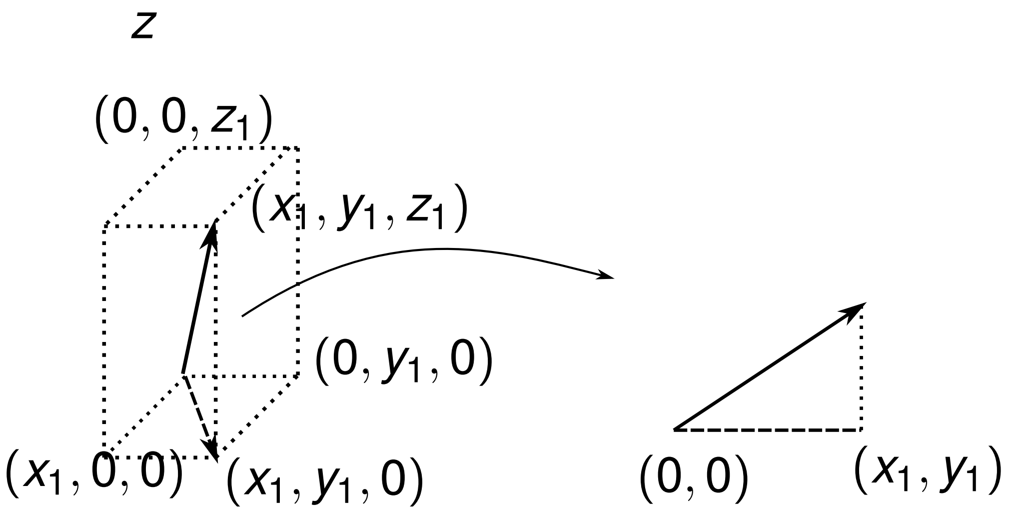

Figure: The triangle with bottom edge in *xy* plane.

Angle between is the same if you rotate both vectors. Rotate until **u** and **v** are in the same *plane*, then use formula from $\mathbb{R}^2$.

Examples

Examples

Example: Let $\mathbf{u} = \langle 3, 1, 3 \rangle$ and $\mathbf{v} = \langle 2, 9, 1 \rangle$. Find $\mathbf{u} \cdot \mathbf{v}$ and use this to compute angle between.

$$u \cdot v = \langle 3, 1, 3 \rangle \cdot \langle 2, 9, 1 \rangle = 3\cdot 2 + 1\cdot 9 + 3\cdot 1 = 6 + 9 + 3 = 18$$

$$|\vec{u}| = \sqrt{3^2 + 1^2 + 3^2} = \sqrt{9 + 1 + 9} = \sqrt{19}$$

$$|\vec{v}| = \sqrt{2^2 + 9^2 + 1^2} = \sqrt{4 + 81 + 1} = \sqrt{86}$$

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos(\theta)$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{18}{\sqrt{19}\sqrt{86}}$$

$$\theta = \cos^{-1}\left(\frac{18}{\sqrt{19}\sqrt{86}}\right) = 63.56^\circ$$

Examples

Example: Let $\mathbf{u} = \langle 3, 1, 3\rangle$ and $\mathbf{v} = \langle 2, 9, 1\rangle$. Find $\mathbf{u}\cdot\mathbf{v}$ and use this to compute angle between.

Exercise: Let $\mathbf{u} = \langle 1, -1, 1\rangle$, $\mathbf{v} = \langle 2, -1, 1\rangle$. Find $\mathbf{u}\cdot\mathbf{v}$ and use this to compute angle between.

$$\vec{u}\cdot\vec{v} = \langle 1,-1,1\rangle \circ \langle 2,-1,1\rangle = 1\cdot 2 + (-1)\cdot(-1) + 1\cdot 1$$

$$= 2+1+1 = 4$$

$$|\vec{u}| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3} \qquad |\vec{v}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{4}{\sqrt{3}\sqrt{6}} = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

$$\theta = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \boxed{19.47^\circ}$$

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Exercise: Let $\mathbf{u} = \langle 1, -1, 1\rangle$, $\mathbf{v} = \langle 2, -1, 1\rangle$. Find $\mathbf{u}\cdot\mathbf{v}$ and use this to compute angle between.

Exercise: Let $\mathbf{u} = \langle x_1, y_1\rangle$. Find a vector $\mathbf{v} \neq \mathbf{0}$ so that $\mathbf{u} \perp \mathbf{v}$.

$\vec{u}\cdot\vec{v} = 0$

$\vec{v} = \langle a, b\rangle$ Then,

$0 = \vec{u}\cdot\vec{v} = x_1 a + y_1 b$ $\Rightarrow$ $-y_1 b = x_1 a$

$\Rightarrow$ $-\dfrac{y_1}{x_1} = \dfrac{a}{b}$

So take $a = -y_1$ $b = x_1$

$\boxed{\vec{v} = \langle -y_1, x_1\rangle}$

(note: there are other answers like $2\vec{v}$)

Properties of dot product

Let $\vec{u} = \langle x_1, y_1 \rangle$ $\vec{v} = \langle x_2, y_2 \rangle$ $\vec{w} = \langle x_3, y_3 \rangle$

Let **u**, **v**, **w** be vectors, and *c* a scalar. Then

commutativity of multiplication

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutative) $\vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2 = x_2x_1 + y_2y_1 = \vec{v}\cdot\vec{u}$
2. $c\mathbf{u} \cdot \mathbf{v} = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ (associative)
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive)

associativity of multiplication.

$$c\vec{u}\cdot\vec{v} = (cx_1)x_2 + (cy_1)y_2 = c(x_1x_2) + c(y_1y_2) = c(u\cdot v)$$

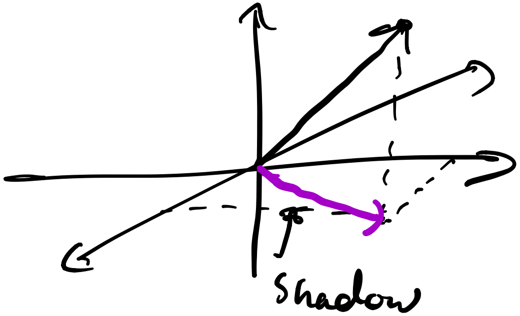
$$= x_1(cx_2) + y_1(cy_2) = u\cdot(cv)$$

$$\vec{u}\cdot(\vec{v}+\vec{w}) = x_1(x_2+x_3) + y_1(y_2+y_3) + z_1(z_2+z_3)$$

$$= x_1x_2 + x_1x_3 + y_1y_2 + y_1y_3 + z_1z_2 + z_1z_3 = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

Projections

Recall: in $\mathbb{R}^3$ we computed length by first computing the length of the “shadow” the vector makes in the *xy* plane.



Projections

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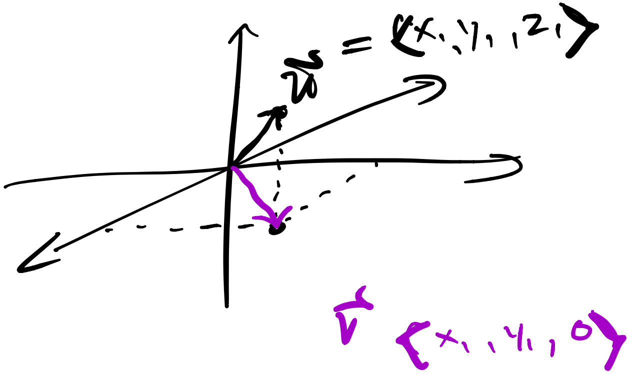
Question: Given two non-zero vectors **u** and **v**, how much of **u** goes in the direction of **v**?



we can think of the shadow example as the amount of $\vec{u}$ in the direction $\vec{v}$

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Question: Given two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, how much of $\mathbf{u}$ goes in the direction of $\mathbf{v}$? What if they are perpendicular?

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Step 1: Compute length of shadow.

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Question: Given two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, how much of $\mathbf{u}$ goes in the direction of $\mathbf{v}$? What if they are perpendicular?
Step 1: Compute length of shadow.
Step 2: Use *direction* of $\mathbf{v}$, or unit vector.

Projections

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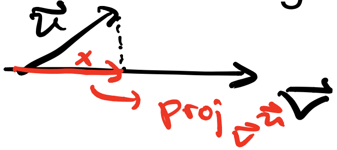
Step 1: Compute length of shadow.

Step 2: Use *direction* of **v**, or unit vector.

Definition: Orthgonal projection of **u** onto **v**:



$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\cdot\frac{\vec{v}}{|\vec{v}|}$$

From precalc $\cos\theta = \frac{x}{|\vec{u}|} \Rightarrow x = |\vec{u}|\cos\theta$

From dot product $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

So, $x = |\vec{u}|\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

length

direction
unit vector
in the $\vec{v}$ direction

Examples

Example: Let $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 4, 0, 1\rangle$. Compute $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$

$$\vec{w}\cdot\vec{v} = \langle 1,2,3\rangle \cdot \langle 4,0,1\rangle = 1\cdot 4 + 2\cdot 0 + 3\cdot 1 = 4+3=7$$

$$|\vec{v}| = \sqrt{4^2+0^2+1^2} = \sqrt{17}$$

$$\Rightarrow \quad \text{proj}_{\vec{v}}\vec{u} = \frac{7}{(\sqrt{17})^2}\langle 4,0,1\rangle = \boxed{\left\langle \frac{28}{17}, 0, \frac{7}{17} \right\rangle}$$

Examples

Example: Let $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 4, 0, 1 \rangle$. Compute $\text{proj}_{\mathbf{v}}\mathbf{u}$.
Exercise: Let $\mathbf{u} = \langle 2, 2, -6 \rangle$ and $\mathbf{v} = \langle 0, 1, 1 \rangle$. Compute $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\vec{u} \cdot \vec{v} = \langle 2, 2, -6 \rangle \cdot \langle 0, 1, 1 \rangle = 2 \cdot 0 + 2 \cdot 1 + -6 \cdot 1$$

$$= 0 + 2 + -6 = -4$$

$$|\vec{v}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{-4}{(\sqrt{2})^2} \langle 0, 1, 1 \rangle = -2 \langle 0, 1, 1 \rangle = \boxed{\langle 0, -2, -2 \rangle}$$

Cross product

Recall:

$$\mathbf{i} = \langle 1, 0, 0 \rangle \,, \ \mathbf{j} = \langle 0, 1, 0 \rangle \,, \ \mathbf{k} = \langle 0, 0, 1 \rangle \,.$$

Cross product

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$$\mathbf{i} = \langle 1, 0, 0 \rangle\,,\ \mathbf{j} = \langle 0, 1, 0 \rangle\,,\ \mathbf{k} = \langle 0, 0, 1 \rangle\,.$$

Definition: Let $\mathbf{u} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2, z_2 \rangle$. The *cross product* is the vector

we go over this later in the slides

$$\mathbf{u} \times \mathbf{v} = (y_1 z_2 - y_2 z_1)\mathbf{i} - (x_1 z_2 - x_2 z_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}.$$

↓ We can think of $\vec{u} \times \vec{v}$ as the "determinant" of

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \vec{i} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{j} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Properties

- Cross product is *perpendicular* to the inputs: $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\perp \mathbf{v}$.

why?

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = (y_1 z_2 - y_2 z_1) x_1 - (x_1 z_2 - x_2 z_1) y_1 + (x_1 y_2 - x_2 y_1) z_1$$

$$= \underline{x_1 y_1 z_2} - \underline{x_1 y_2 z_1} - \underline{x_1 y_1 z_2} + \underline{y_1 x_2 z_1} + \underline{x_1 y_2 z_1} - \underline{x_2 y_1 z_1}$$

$$= 0.$$

Properties

- Cross product is *perpendicular* to the inputs: $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\perp \mathbf{v}$. Can prove this directly from formula of $\mathbf{u} \times \mathbf{v}$.
- Anti-commutative

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

Properties

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- Anti-commutative

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

$$\implies \mathbf{u} \times \mathbf{u} = \mathbf{0}.$$

since $\vec{u} \times \vec{u} = -\vec{u} \times \vec{u}$

$\Rightarrow 2(\vec{u} \times \vec{u}) = \vec{0}$

$\Rightarrow \vec{u} \times \vec{u} = \vec{0}.$

- Associative:

$$(a\mathbf{u}) \times (b\mathbf{v}) = (ab)(\mathbf{u} \times \mathbf{v})$$

Properties

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- Associative:

$$(a\mathbf{u}) \times (b\mathbf{v}) = (ab)(\mathbf{u} \times \mathbf{v})$$

- Distributive:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

Determinant

Several ways to remember: *determinant*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}$$

Where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. so $\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} = y_1 z_2 - y_2 z_1$

Another way to remember

Right hand rule! And unit vectors.

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Right hand rule! And unit vectors. Cross product of two unit vectors is the third unit vector with sign determined by right hand rule:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

means **i** first, then **j**, and your thumb points in the third direction.

Another way to remember

Right hand rule! And unit vectors. Cross product of two unit vectors is the third unit vector with sign determined by right hand rule:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

means **i** first, then **j**, and your thumb points in the third direction. Similarly

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \ \ \mathbf{j} \times \mathbf{k} = \mathbf{i}.$$

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means **i** first, then **j**, and your thumb points in the third direction. Similarly

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \ \mathbf{j} \times \mathbf{k} = \mathbf{i}.$$

Opposite order changes sign!

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

etc.

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$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \;\; \mathbf{j} \times \mathbf{k} = \mathbf{i}.$$

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etc.
Note:

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}.$$

Cross product with unit vectors

Write

$$\langle x_1, y_1, z_1\rangle = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}, \ \langle x_2, y_2, z_2\rangle = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

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we use the distributive property

$$\begin{aligned}
&(\underline{x_1\mathbf{i}} + \underline{y_1\mathbf{j}} + z_1\mathbf{k}) \times (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})\\
&\quad = \underline{x_1x_2\mathbf{i}\times\mathbf{i} + x_1y_2\mathbf{i}\times\mathbf{j} + x_1z_2\mathbf{i}\times\mathbf{k}}\\
0 \rightarrow &\qquad + \underline{y_1x_2\mathbf{j}\times\mathbf{i} + y_1y_2\mathbf{j}\times\mathbf{j} + y_1z_2\mathbf{j}\times\mathbf{k}}\\
0\ &\qquad + \underline{z_1x_2\mathbf{k}\times\mathbf{i} + z_1y_2\mathbf{k}\times\mathbf{j} + z_1z_2\mathbf{k}\times\mathbf{k}} \leftarrow 0\\
&\quad = x_1y_2\mathbf{k} + x_1z_2(-\mathbf{j}) + y_1x_2(-\mathbf{k}) + y_1z_2\mathbf{i} + z_1x_2\mathbf{j} + z_1y_2(-\mathbf{i})\\
&\quad = (y_1z_2 - z_1y_2)\mathbf{i} - (x_1z_2 - z_1x_2)\mathbf{j} + (x_1y_2 - y_1x_2)\mathbf{k}.
\end{aligned}$$

Note sign on **j**!

Examples

Example: Let $\mathbf{u} = \langle 2, -1, -1 \rangle$ and $\mathbf{v} = \langle 1, 0, 3 \rangle$. Compute $\mathbf{u} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ 1 & 0 & 3 \end{vmatrix} = \vec{i}\left(-1\cdot 3 - (0\cdot -1)\right) - \vec{j}\left(2\cdot 3 - (1\cdot -1)\right) + \vec{k}\left(2\cdot 0 - (1\cdot -1)\right)$$

$$= -3\vec{i} - 7\vec{j} + \vec{k}$$

$$= \langle -3, -7, 1 \rangle$$

Examples

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Exercise: Let $\mathbf{u} = \langle 3, 1, 2 \rangle$ and $\mathbf{v} = \langle 6, -3, 1 \rangle$. Compuate $\mathbf{v} \times \mathbf{u}$ (note the order).

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

answer on next slide

Examples

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Answer:

$$(-6 - 1)\mathbf{i} - (12 - 3)\mathbf{j} + (6 + 9)\mathbf{k} = \langle -7, -9, 15 \rangle .$$

Another formula

Theorem: Let **u** and **v** be non-zero vectors. Then the *length of the cross product* is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where θ is the angle between **u** and **v**.

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Proof of *direction*: use **i**, **j** and **k** - these use right hand rule for cross product.

Proof of length

Rough idea: rotate so that **u** lies along *x* axis

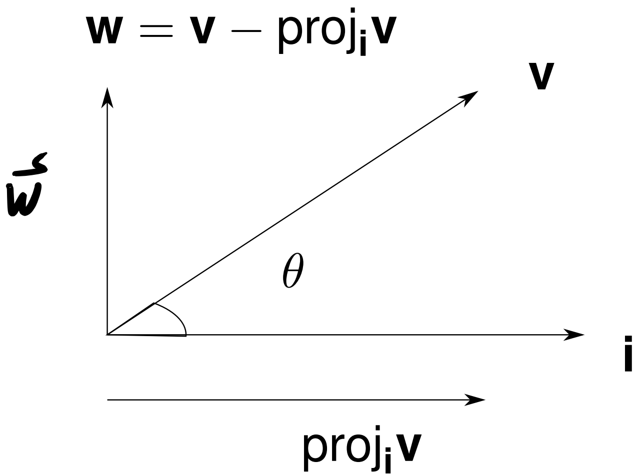

Figure: Unit vectors and the cross product.

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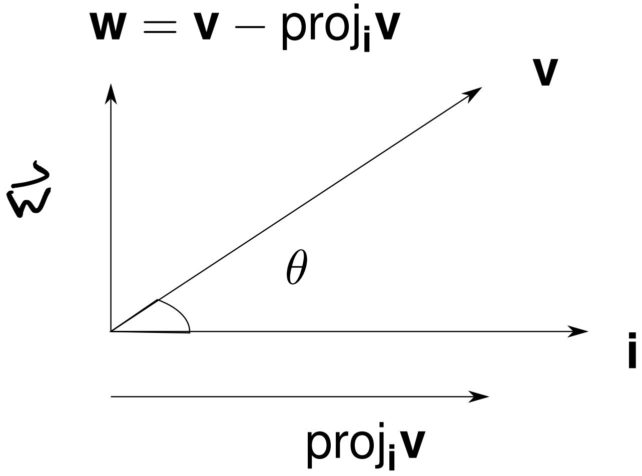

Figure: Unit vectors and the cross product.

$|\mathbf{u}|$ and $|\mathbf{v}|$ scale out in formula, so can prove for unit vectors:
$\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \langle a, b, c\rangle$ with $(a^2 + b^2 + c^2)^{1/2} = 1$.

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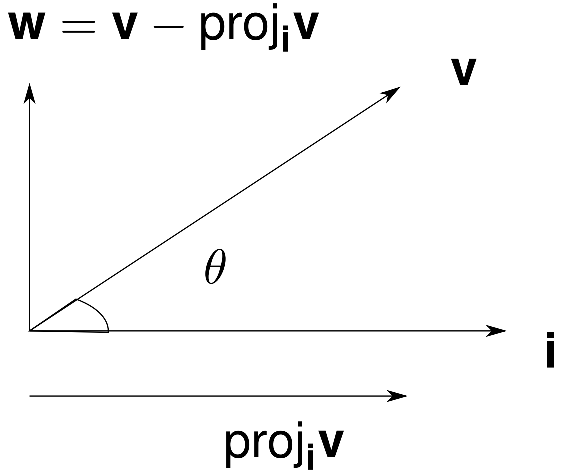

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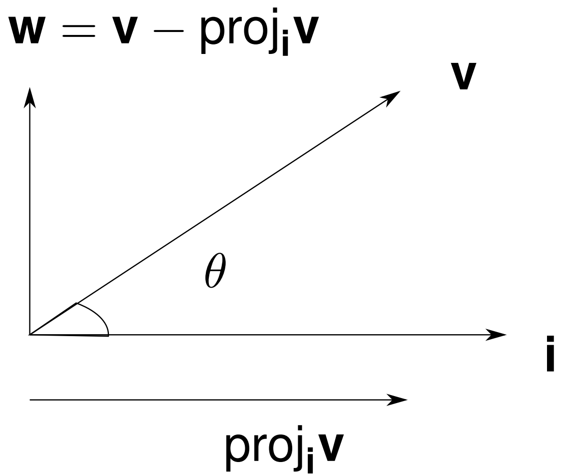

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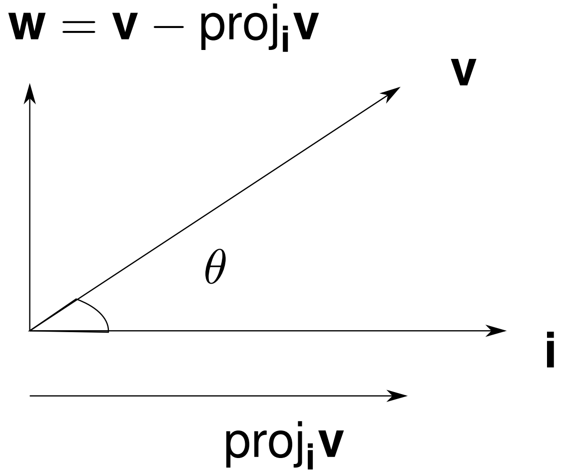
Proof of length

Rough idea: rotate so that $\mathbf{u}$ lies along x axis



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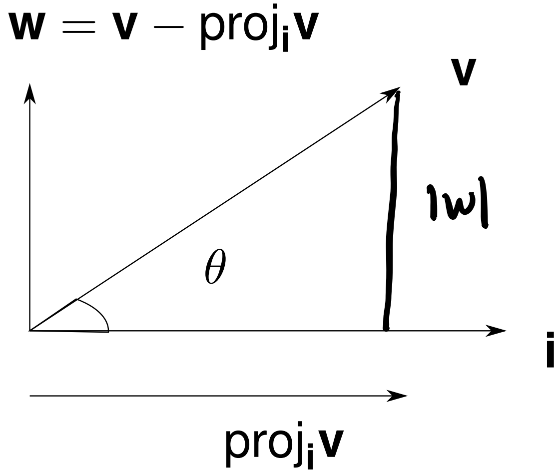
$\rightsquigarrow |\mathbf{w}| = |\mathbf{i} \times \mathbf{v}|$.

Proof of length

Rough idea: rotate so that **u** lies along x axis



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$\rightsquigarrow$ $|\mathbf{w}| = |\mathbf{i} \times \mathbf{v}|$. Trig $\rightsquigarrow$ $|\mathbf{w}| = \sin\theta$. $\sin\theta = \frac{|\vec{w}|}{|\vec{v}|} = |w|$ since $|\vec{v}| = 1$

Application: area of a parallelogram

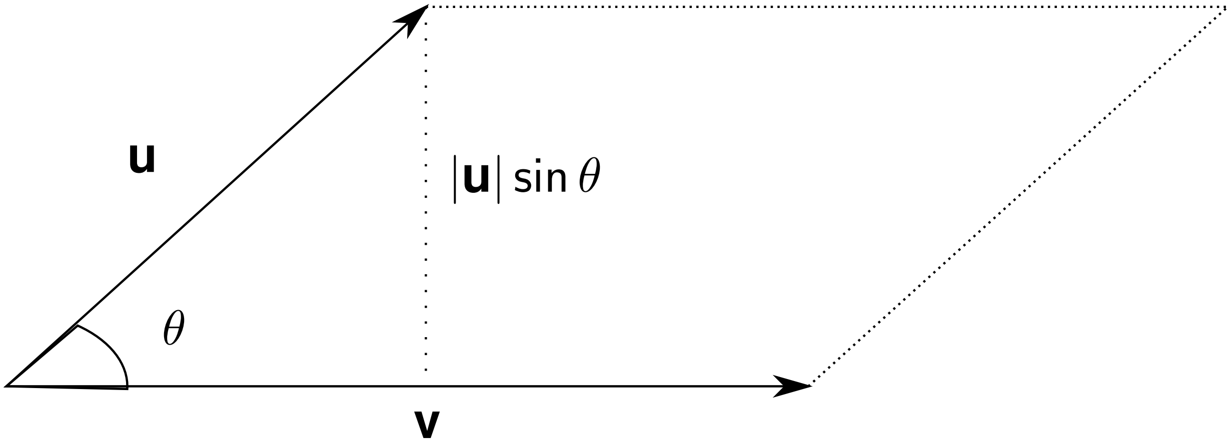
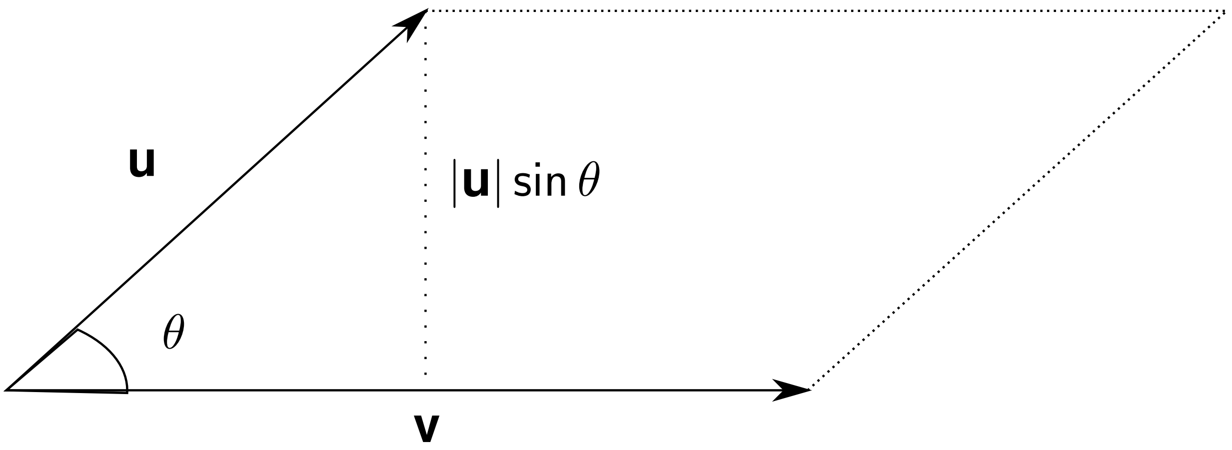


Figure: The parallelogram *spanned* by **u** and **v**.

Application: area of a parallelogram



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Area is length of bottom times height:

$$\text{Area} = |\mathbf{v}||\mathbf{u}|\sin\theta = |\mathbf{v}\times\mathbf{u}|.$$

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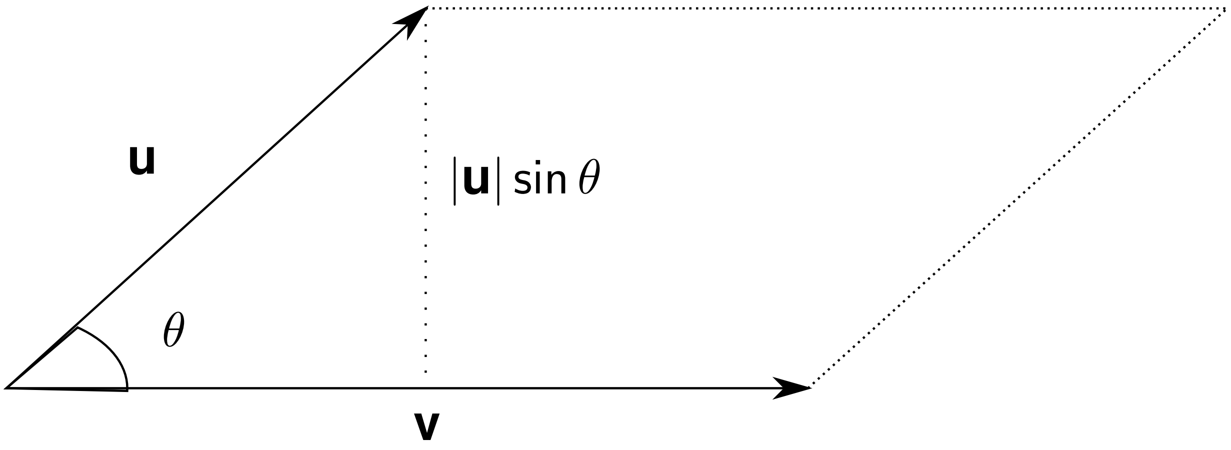

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Area is length of bottom times height:

$$\text{Area} = |\mathbf{v}||\mathbf{u}|\sin\theta = |\mathbf{v}\times\mathbf{u}|.$$

What about $|\mathbf{u}\times\mathbf{v}|$? Note that $|\vec{u}\times\vec{v}| = |-(\vec{v}\times\vec{u})|$
$= |\vec{v}\times\vec{u}|.$

Application: area of a triangle

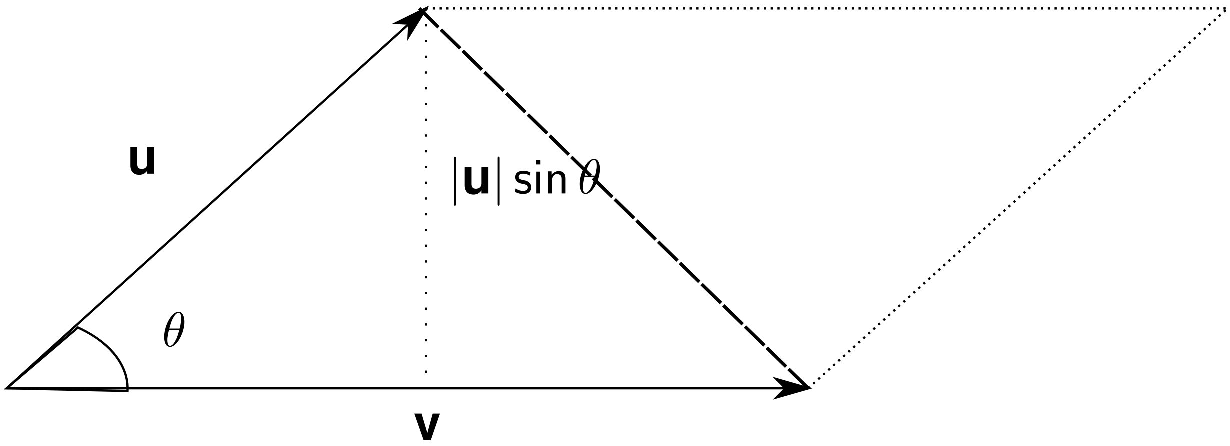

Figure: The triangle *spanned* by **u** and **v** has area $\frac{1}{2}$ the area of parallelogram.

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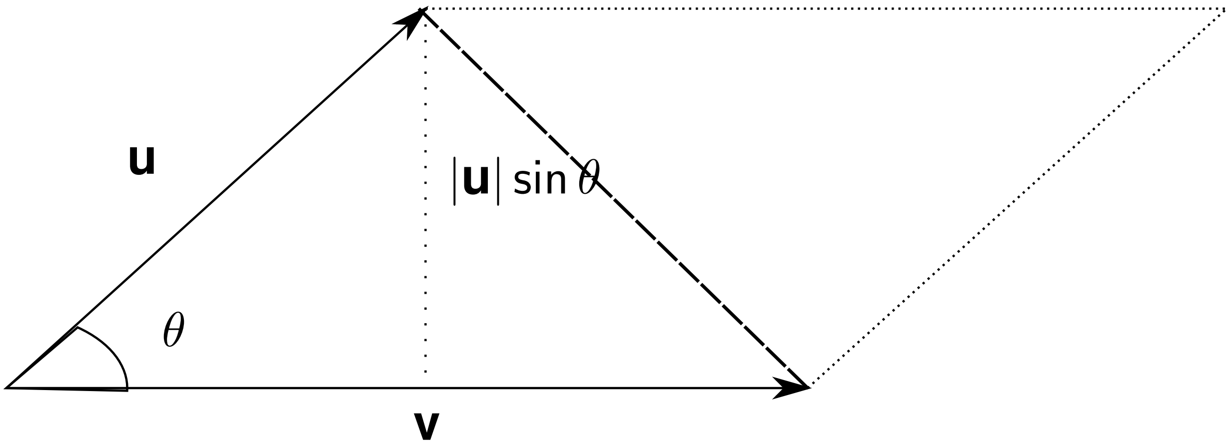

Figure: The triangle *spanned* by **u** and **v** has area $\frac{1}{2}$ the area of parallelogram.

$$\text{Area} = \frac{1}{2}|\mathbf{u} \times \mathbf{v}|.$$