# Green’s Theorem (17.4)

July 17, 2020

# Big Picture for today

**Big Picture:** Green's Theorem relates a line integral over a closed path to an area integral on the interior of the path.

# Oriented curves

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Change direction: $u = 1 - t \rightsquigarrow \mathbf{r}(u) = \langle 2(1-u), 1-u \rangle$, $du = -dt$. so

$$\int_C f ds = \int_1^0 2(1-u)(1-u)\sqrt{5}(-du) = \int_0^1 (2 - 4u + 2u^2)\sqrt{5} du$$
$$= (2/3)\sqrt{5}$$

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This means that a scalar line integral does not depend on direction, but a vector line integral changes sign.

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**Definition:** A curve $C$ parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$ is *closed* if $\mathbf{r}(b) = \mathbf{r}(a)$. $C$ is *simple* if there are no self intersections. A closed curve $C$ is *oriented* if the parametrization goes around $C$ counter-clockwise.

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**Notation:** If $C$ is a simple closed curve, we sometimes write $\oint_C$ for $\int_C$.

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$f = x$, $g = x$ so $g_x - f_y = 1$, so

$$\iint_R (g_x - f_y) dA = \iint_R (1) dA = 1/2$$

is just the area of the triangle.

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$$\int_C \mathbf{F} \cdot d\mathbf{r} = 6.$$

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On the other hand, we have $f = -3y$ and $g = 3x$, so $g_x - f_y = 6$.

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$$\iint_T (g_x - f_y)dA = \int_0^1 \int_0^{2-2x} 6dydx = 6.$$

# Example

Let $\mathbf{F} = \left\langle 2y + \cos(x^4), -2x - e^{y^2} \right\rangle$, and let $C$ be the curve starting with the segment from $(0, 0)$ to $(\pi, 0)$, followed by the part of the curve $y = \sin x$ starting at $(\pi, 0)$ and returning to $(0, 0)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

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Let $R$ be the region enclosed by $C$, $0 \leq x \leq \pi$, $0 \leq y \leq \sin x$.

$f = 2y + \cos(x^4)$, $g = -2x - e^{y^2}$, so $g_x - f_y = -4$.

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$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (g_x - f_y) dA = \int_0^{\pi} \int_0^{\sin x} (-4) dy dx$$
$$= -4 \int_0^{\pi} \sin x dx = -4(-\cos x)|_0^{\pi} = -8.$$

# Exercise

Let $\mathbf{F} = \langle xy^2 + xy + (1 + x^2)^{-1/4}, \ln(8(1 + 2y^4)) + x^2 \rangle$. Let $C$ be the segment from $(0,0)$ to $(1,0)$, followed by the quarter circle arc in the upper half plane starting at $(1,0)$ and ending at $(0,1)$, followed by the segment from $(0,1)$ to $(0,0)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

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Let $R$ be the region enclosed by $C$: $0 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$.

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$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (g_x - f_y)\,dA \\
&= \int_0^1 \int_0^{\sqrt{1-x^2}} (g_x - f_y)\,dy\,dx \\
&= \int_0^1 \int_0^{\sqrt{1-x^2}} (2x - (2xy + x))\,dy\,dx \\
&= \int_0^1 (xy - xy^2)\Big|_0^{\sqrt{1-x^2}} dx \\
&= \int_0^1 (x(1-x^2)^{1/2} - x(1-x^2))\,dx = 1/3 - 1/4.
\end{aligned}
$$

# Example

Let $\mathbf{F} = \langle xe^y, x \rangle$, and let $C$ be the boundary of the domain enclosed by $y = x^2$, $x = 2$, and the $x$ axis with counter-clockwise orientation. Use Green's theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

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$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \int_0^{x^2} (1 - xe^y)\,dy\,dx \\
&= \int_0^2 (x^2 - xe^{x^2} + x)\,dx \\
&= (x^3/3 - \frac{1}{2}e^{x^2} + \frac{1}{2}x^2)\Big|_0^2 \\
&= (8/3 - \frac{1}{2}e^4 + 2) - (0 - \frac{1}{2} + 0).
\end{aligned}$$